



King's Research Portal

DOI:

[10.1016/j.spa.2016.07.011](https://doi.org/10.1016/j.spa.2016.07.011)

Document Version

Peer reviewed version

[Link to publication record in King's Research Portal](#)

Citation for published version (APA):

Gobet, E., & Turkedjiev, P. (2016). Adaptive importance sampling in least-squares Monte Carlo algorithms for backward stochastic differential equations. *Stochastic Processes and Their Applications*.
<https://doi.org/10.1016/j.spa.2016.07.011>

Citing this paper

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

General rights

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Accepted Manuscript

Adaptive importance sampling in least-squares Monte Carlo algorithms for backward stochastic differential equations

E. Gobet, P. Turkedjiev

PII: S0304-4149(16)30123-5

DOI: <http://dx.doi.org/10.1016/j.spa.2016.07.011>

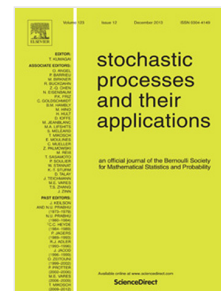
Reference: SPA 3006

To appear in: *Stochastic Processes and their Applications*

Received date: 27 June 2015

Revised date: 9 July 2016

Accepted date: 25 July 2016



Please cite this article as: E. Gobet, P. Turkedjiev, Adaptive importance sampling in least-squares Monte Carlo algorithms for backward stochastic differential equations, *Stochastic Processes and their Applications* (2016), <http://dx.doi.org/10.1016/j.spa.2016.07.011>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

Adaptive importance sampling in least-squares Monte Carlo algorithms for backward stochastic differential equations *

E. Gobet^{†‡}

Centre de Mathématiques Appliquées
Ecole Polytechnique and CNRS
Route de Saclay
F 91128 Palaiseau cedex, France

P. Turkedjiev[§]

Department of Mathematics
King's College London
Strand
London WC2R 2LS, U.K.

August 19, 2016

Abstract

We design an importance sampling scheme for backward stochastic differential equations (BSDEs) that minimizes the conditional variance occurring in least-squares Monte-Carlo (LSMC) algorithms. The Radon-Nikodym derivative depends on the solution of BSDE, and therefore it is computed adaptively within the LSMC procedure. To allow robust error estimates w.r.t. the unknown change of measure, we properly randomize the initial value of the forward process. We introduce novel methods to analyze the error: firstly, we establish norm stability results due to the random initialization; secondly, we develop refined concentration-of-measure techniques to capture the variance reduction. Our theoretical results are supported by numerical experiments.

Keywords: Backward stochastic differential equations, empirical regressions, importance sampling

MSC Classification: 49L20, 60H07, 62Jxx, 65C30, 93E24

1 Introduction

Importance sampling can be important for accelerating the convergence of Monte-Carlo approximation. To name a few examples and references, it has applications in numerical integration [32, 22, 25] and in rare event simulation [9, 31, 7]. The idea is to direct the simulations

*This research is part of the Chair Financial Risks of the Risk Foundation and of the FiME Laboratory.

[†]Email: emmanuel.gobet@polytechnique.edu

[‡]This work is part of the Chair Financial Risks of the Risk Foundation, the Finance for Energy Market Research Centre and the ANR project CAESARS (ANR-15-CE05-0024)

[§]Email: plamen.turkedjiev@kcl.ac.uk

to important regions of space through an appropriate change of measure. In this work, we propose a fully implementable least-squares Monte-Carlo (LSMC, a.k.a. Regression Monte-Carlo or Empirical Regression) scheme with importance sampling for Backward Stochastic Differential Equations (BSDEs). BSDEs are usually associated to stochastic control problems [27, 12]. In the Markovian case with fixed terminal time $T > 0$, the BSDE takes the form

$$\mathcal{Y}_t = g(X_T^{(R)}) + \int_t^T f(s, X_s^{(R)}, \mathcal{Y}_s, \mathcal{Z}_s) ds - \int_t^T \mathcal{Z}_s dW_s \quad (1)$$

where the unknown processes are $(\mathcal{Y}, \mathcal{Z})$ and $X^{(R)}$ is a given forward (d -dimensional) SDE initialized by a random variable R and driven by the (q -dimensional) Brownian motion W :

$$X_t^{(R)} = R + \int_0^t b(s, X_s^{(R)}) ds + \int_0^t \sigma(s, X_s^{(R)}) dW_s. \quad (2)$$

In the recent years, there have been many contributions to their numerical approximations (see references in [6, 8, 10, 19, 18]). Loosely speaking, the current work extends [18] by incorporating an importance sampling that minimizes the variance of the LSMC algorithm for computing the value functions y and z – defined as $\mathcal{Y}_t = y(t, X_t^{(R)})$ and $\mathcal{Z}_t = z(t, X_t^{(R)})$ – using a suitable Dynamic Programming Equation (DPE for short). The value functions y and z do not depend on the random variable R , but only on the transition density of the SDE. The flexibility in the choice of R gives a degree of freedom which we will take advantage of.

The work closest to ours is the one by Bender and Moseler [5] where they propose an importance sampling method for BSDEs for a general (but known) Radon-Nikodym derivative. In our setting, the change of measure is not given but sought within the algorithm. Indeed, the optimal Radon-Nikodym derivative depends on the solution $(\mathcal{Y}, \mathcal{Z})$ – see Proposition 2.1 – therefore it too must be approximated. In fact, the change of probability measure implies that traditional simulation and error analysis techniques cannot be applied to establish convergence of the approximations of y and z . In [18], the simulations of $X^{(R)}$ for the LSMC scheme were generated from a fixed point at time 0. The error was analyzed using the L_2 -norm related to the law of simulations. Propagation of error terms due to dynamic programming were treated using BSDE estimates and an extension of Gronwall's inequality. In importance sampling, the simulations used in LSMC scheme have a modified drift because of the change of probability measure. The drift term depends on the solution of the BSDE or its approximation. This creates two novel difficulties. Firstly, one cannot generate the simulations from time 0 with a fixed dynamics. Indeed, the drift change depends on the BSDE solution, which is computed recursively backwards in time, and therefore not available for simulation from time zero. To overcome this issue, we initialize the simulations at each time point i using a generic random variable. However, in doing so, in general we lose the ability to treat the DPE error with the usual Gronwall technique. To retrieve the algorithm convergence, the initializing distribution is required to satisfy the Uniform Sub-Exponential Sandwiching (USES) property, see

Proposition 3.1. This will determine the choice of R used in (2). This allows the distribution of simulations to have equivalent \mathbf{L}_2 -norms, under different changes of measure and at every time-point. Secondly, the error due to the approximation of the Radon-Nikodym derivative must be treated.

Another significant contribution of this work is in the field of nonparametric statistics and it is crucial for capturing the effect of variance reduction in our scheme. In [18], the error analysis makes use of classical regression theory [20, Theorems 11.1 and 11.3]. In this theory, error estimates are not sufficiently tight to observe impact of variance reduction. In particular, uniform concentration-of-measure techniques are used, which only observe the diameter of the approximation space and cannot observe low variance. We make a non-trivial continuation of the recent work of [4] in order to improve concentration-of-measure techniques in the case of regression problems, so that we can recover the variance reduction effect. These results are novel, to the best of our knowledge, and may bring insights in problems beyond the immediate concern of BSDE approximation.

Overview. In Section 2.1, we identify the optimal importance sampling probability measure starting from the continuous time BSDE (1). In Section 2.2, we introduce the discrete-time approximation of (1) in the form of the importance sampling DPEs. We also state the assumptions on the data g and f , and summarize some key properties of the resulting discrete time BSDE. In Section 2.3, we present the full regression scheme combined with adaptive importance sampling (called ISMWDP scheme). In Section 3, we define USES and give nontrivial, fully implementable examples (Proposition 3.3) relevant to practical problems. In Section 4, we detail the regression scheme (LSMC) on piecewise constant basis functions in a general setting (unrelated to BSDEs). We provide explicit nonparametric error estimates which do not require uniform concentration-of-measure techniques (Theorem 4.1). In Section 5, we analyse the convergence of the ISMWDP scheme. Explicit error estimates for this fully implementable scheme are derived. We conclude the paper in Section 6 with numerical examples that illustrate the performance of the scheme.

Model restrictions. Due to the novel nature of this scheme, we make simplified assumptions on the BSDE model and on the numerical method, with the aim of highlighting the main ideas rather than technical results. In particular, we assume that the function $f(t, x, y, z) \equiv f(t, x, y)$ is independent of the z component. Of course considering such an f is a restriction for applications, but nonetheless it still serves a significant interest since it allows to handle reaction-diffusion equations [33] and nonlinear valuations in finance [11]. We discuss more serious reasons for this simplification in Subsection 2.5, in the hope to motivate further investigation.

2 Derivation of the importance sampling algorithm

2.1 Derivation of optimal Radon-Nikodym derivative

We are concerned with BSDEs driven by a q -dimensional Brownian motion W , supported by a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with the usual conditions on the filtration. The solution $(\mathcal{Y}, \mathcal{Z})$ is progressively measurable, takes values in $\mathbb{R} \times (\mathbb{R}^q)^\top$, and satisfies

$$\mathcal{Y}_t = \xi + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) ds - \int_t^T \mathcal{Z}_s dW_s \quad (3)$$

with the process \mathcal{Y} being continuous; the operator $^\top$ denotes the vector transpose, i.e. \mathcal{Z} is a row vector. Here, the BSDE is not necessarily of Markovian type.

Before introducing our numerical scheme, we first identify the optimal change of measure. This is inspired by the continuous time equation (3). Assume for discussion that the solution $(\mathcal{Y}, \mathcal{Z})$ of (3) is unique in \mathbf{L}_2 -spaces, so that the above solutions satisfy

$$\mathbb{E} \left[\sup_{t \leq T} |\mathcal{Y}_t|^2 + \int_0^T |\mathcal{Z}_t|^2 dt \right] < +\infty;$$

this is valid under fairly general conditions on the data ξ and f [27, 12]. We start from the representation of \mathcal{Y} as conditional expectation under \mathbb{P} :

$$\mathcal{Y}_t = \mathbb{E}_{\mathbb{P}} \left[\xi + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) ds \mid \mathcal{F}_t \right].$$

Let h be a $(\mathbb{R}^q)^\top$ -valued progressively measurable process, such that one can define the process $(W_t^{(h)})_{0 \leq t \leq T}$ and the \mathbb{P} -martingale $(1/\mathcal{L}_t^{(h)})_{0 \leq t \leq T}$ by the stochastic exponential of $\int_0^\cdot h_s dW_s$:

$$W_t^{(h)} := W_t - \int_0^t h_r^\top dr, \quad \mathcal{L}_t^{(h)} := e^{-\int_0^t h_r dW_r + \frac{1}{2} \int_0^t |h_r|^2 dr} = e^{-\int_0^t h_r dW_r^{(h)} - \frac{1}{2} \int_0^t |h_r|^2 dr}, \quad (4)$$

$$\mathcal{L}_{t,s}^{(h)} := \frac{\mathcal{L}_s^{(h)}}{\mathcal{L}_t^{(h)}} \quad \text{for } 0 \leq t \leq s \leq T.$$

Assuming that $\int_0^\cdot h_s dW_s$ is a BMO martingale [23], we can set a new equivalent measure $\mathbb{Q}^h \sim \mathbb{P}$ with Radon-Nikodym derivative $(1/\mathcal{L}_T^{(h)})$, so that

$$\frac{d\mathbb{Q}^h}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = [\mathcal{L}_t^{(h)}]^{-1}, \quad \text{and} \quad \frac{d\mathbb{P}}{d\mathbb{Q}^h} \Big|_{\mathcal{F}_t} = \mathcal{L}_t^{(h)}.$$

Under \mathbb{Q}^h , $W^{(h)}$ is a Brownian motion by Girsanov's theorem, and $(\mathcal{L}_{t,s}^{(h)})_{s \geq t}$ is a \mathbb{Q}^h -martingale, being the stochastic exponential of $\int_t^\cdot (-h_s) dW_s^{(h)}$. Moreover, we can express \mathcal{Y} under the new

measure

$$\mathcal{Y}_t = \frac{\mathbb{E}_{\mathbb{Q}^h} \left[\xi \mathcal{L}_T^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_s^{(h)} ds \mid \mathcal{F}_t \right]}{\mathbb{E}_{\mathbb{Q}^h} \left[\mathcal{L}_T^{(h)} \mid \mathcal{F}_t \right]} = \mathbb{E}_{\mathbb{Q}^h} \left[\xi \mathcal{L}_{t,T}^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_{t,s}^{(h)} ds \mid \mathcal{F}_t \right]. \quad (5)$$

To obtain variance reduction in the Monte-Carlo based algorithm, the \mathcal{F}_t -conditional variance under \mathbb{Q}^h of $S(t, h) := \xi \mathcal{L}_{t,T}^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_{t,s}^{(h)} ds$ has to be small; we see this later in Corollary 4.2. Under the assumptions

(H_a) there exists $\varepsilon > 0$ such that $\mathbb{P}(\forall t \in [0, T] : \mathcal{Y}_t \geq \varepsilon) = 1$,

(H_b) $\int_0^\cdot \mathcal{Z}_s dW_s$ is a BMO martingale,

we determine an optimal Radon-Nikodym derivative in the following proposition.

Proposition 2.1. *Assume the existence of a unique solution $(\mathcal{Y}, \mathcal{Z})$ to (3) which satisfies $\mathbb{E}(\sup_{t \leq T} |\mathcal{Y}_t|^2 + \int_0^T |\mathcal{Z}_t|^2 dt) < +\infty$ and (H_a)–(H_b). Then the equivalent probability measure \mathbb{Q}^h such that the random variable $S(t, h)$ has zero \mathcal{F}_t -conditional variance in (5) is given by the drift $h_s = \mathcal{Z}_s / \mathcal{Y}_s$.*

The above result extends [30] to BSDEs.

Remark 2.2. The assumptions (H_a)–(H_b) can be satisfied under quite general conditions on the data ξ and f . In fact, even if (H_a) is not immediately satisfied, a trivial transformation may permit the application of Proposition 2.1: if $\mathcal{Y}_t \geq c$ for any $t \in [0, T]$ a.s. for some possibly non-positive constant $c \in \mathbb{R}$, the shifted BSDE $(\mathcal{Y} - c + 1, \mathcal{Z})$ associated to data $\xi - c + 1$ and $f(t, y + c - 1, z)$ satisfies the assumption (H_a) with $\varepsilon = 1$; likewise, if $\mathcal{Y}_t \leq c$, one can apply similar arguments with an additional sign flip of \mathcal{Y} . The BMO-condition (H_b) is satisfied in many situations, in particular when the terminal condition ξ is bounded [3].

Proof of Proposition 2.1. Let h be a progressively measurable process such that $\int_0^\cdot h_s dW_s$ is a BMO martingale. Then the stochastic exponential $1/\mathcal{L}^{(h)}$ is an uniformly integrable martingale under \mathbb{P} [23, Theorem 2.3]. By Itô's formula applied to $\mathcal{Y} \mathcal{L}^{(h)}$ on $[t, T]$ combined with (3), we readily obtain

$$\begin{aligned} S(t, h) &= (\mathcal{L}_t^{(h)})^{-1} \left(\mathcal{Y}_T \mathcal{L}_T^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_s^{(h)} ds \right) \\ &= \mathcal{Y}_t + (\mathcal{L}_t^{(h)})^{-1} \int_t^T \left(\mathcal{L}_s^{(h)} [-f(s, \mathcal{Y}_s, \mathcal{Z}_s) ds + \mathcal{Z}_s dW_s] + \mathcal{Y}_s \mathcal{L}_s^{(h)} (-h_s) dW_s^{(h)} - \mathcal{L}_s^{(h)} h_s \cdot \mathcal{Z}_s ds \right) \\ &\quad + (\mathcal{L}_t^{(h)})^{-1} \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_s^{(h)} ds \\ &= \mathcal{Y}_t + (\mathcal{L}_t^{(h)})^{-1} \int_t^T \mathcal{L}_s^{(h)} (\mathcal{Z}_s - \mathcal{Y}_s h_s) dW_s^{(h)}. \end{aligned}$$

(\mathbf{H}_a) – (\mathbf{H}_b) readily ensure $h_s = \mathcal{Z}_s/\mathcal{Y}_s$ is well defined, and that $\int_0^\cdot h_s dW_s$ is a BMO martingale. Therefore, $1/\mathcal{L}^{(h)}$ is a uniformly integrable martingale [23, Theorem 2.3], and the resulting equivalent probability measure \mathbb{Q}^h is well defined from Girsanov’s theorem. Moreover, $S(t, h)$ has zero \mathcal{F}_t -conditional variance under \mathbb{Q}^h . \square

2.2 Assumptions and discrete-time scheme

The resolution of $(\mathcal{Y}, \mathcal{Z})$ usually requires the time-discretization of (3). A possible approach is to use the Malliavin integration-by-parts formula to represent \mathcal{Z} [28], which leads to the Malliavin weights DPE [18, 34] (MWDP) given by

$$\text{for } i \in \{0, \dots, N-1\}, \quad \begin{cases} Y_i &= \mathbb{E}_{\mathbb{P}, i} \left[\xi + \sum_{k=i}^{N-1} f(t_k, Y_{k+1}) \Delta_k \right], \\ Z_i &= \mathbb{E}_{\mathbb{P}, i} \left[\xi \Theta_N^{(i)} + \sum_{k=i+1}^{N-1} f(t_k, Y_{k+1}) \Theta_k^{(i)} \Delta_k \right] \end{cases} \quad (6)$$

where (Y_i, Z_i) approximates $(\mathcal{Y}_{t_i}, \mathcal{Z}_{t_i})$ along the time-grid $\pi = \{0 := t_0 < \dots < t_i < \dots < t_N := T\}$, $\Delta_i := t_{i+1} - t_i$, $\mathbb{E}_{\mathbb{P}, i}[\cdot] := \mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{F}_{t_i}]$, and where $\{\Theta_k^{(i)} : k = i+1, \dots, N\}$ is a suitable sequence of square integrable random variables.¹ For reasons detailed in Section 2.5, we consider functions f that are independent of z from here on; this is not the case of [18]. It is important to assume that, for all i and k , $\Theta_k^{(i)}$ satisfies

$$\mathbb{E}_{\mathbb{P}, i} [\Theta_k^{(i)}] = 0, \quad \mathbb{E}_{\mathbb{P}, i} [|\Theta_k^{(i)}|^2]^{1/2} \leq \frac{C_M}{\sqrt{t_k - t_i}},$$

for some finite constant C_M .

Remark. In fact, the MWDP is known [34] to be a ‘good’ discrete time approximation of (3) in the Markovian framework (1). We choose a DPE based on Malliavin weights since we know from [18] that it allows better control on the variance terms in the convergence analysis compared to alternative schemes which take $\Theta_k^{(i)} = \Delta W_k / \Delta_k$ for all i , see [19][15][26][29]. Nevertheless, the subsequent importance sampling scheme could be designed with other DPEs, and this would not greatly affect the arguments of this section which follow.

We now introduce the discrete-time importance sampling scheme on which we base the LSMC scheme later. As will be explained in Subsection 2.5, we assume that f does not depend on z . We first define simplified notation to deal with the discrete-time counterpart of the importance sampling inverse Radon-Nikodym derivative (4). For a given piecewise constant (in time), bounded, adapted process h with $h|_{(t_k, t_{k+1}]} := h_k \in (\mathbb{R}^q)^\top$ \mathcal{F}_{t_k} -measurable, let

$$L_j^{(h)} := \exp \left(- \sum_{k=0}^{j-1} h_k \Delta W_k + \frac{1}{2} \sum_{k=0}^{j-1} |h_k|^2 \Delta_k \right), \quad 0 \leq j \leq N,$$

¹These random variables are so-called Malliavin weights, from which the scheme gets its name. They are derived from Malliavin integration-by-parts formula.

where we write $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. We also define

$$L_{i,j}^{(h)} := \frac{L_j^{(h)}}{L_i^{(h)}} = \exp \left(- \sum_{k=i}^{j-1} h_k \Delta W_k + \frac{1}{2} \sum_{k=i}^{j-1} |h_k|^2 \Delta_k \right).$$

As indicated in Proposition 2.1, h will be computed backward in time using the processes (Y, Z) . In principle, we would now set $h_k = Z_k/Y_k$ and define $d\mathbb{Q}^h|_{\mathcal{F}_{t_i}}/d\mathbb{P}|_{\mathcal{F}_{t_i}} = [L_i^{(h)}]^{-1}$ and

$$Y_i = \mathbb{E}_{\mathbb{Q}^h, i} \left[\xi L_{i,N}^{(h)} + \sum_{k=i}^{N-1} f(t_k, Y_{k+1}) L_{i,k+1}^{(h)} \Delta_k \right],$$

as we did in Section 2.1. However, at time point i , the solution (Y_i, Z_i) is not known explicitly, therefore it isn't possible to compute $L_{i,k+1}^{(h)}$ explicitly. In order to have an explicit formulation, we use a modified probability measure defined as follows: for $0 \leq i, j \leq N$, set

$$\mathbb{Q}_i^h|_{\mathcal{F}_{t_j}} := \left([L_{i+1,j}^{(h)}]^{-1} \mathbf{1}_{j>i+1} + \mathbf{1}_{j \leq i+1} \right) \mathbb{P}|_{\mathcal{F}_{t_j}}.$$

Since h is bounded, the Girsanov theorem implies that \mathbb{Q}_i^h is a probability measure. Observe that the change of measure is effective from time $i+2$ instead of $i+1$. We denote by $W^{\mathbb{Q}_i^h}$ the new Brownian motion under \mathbb{Q}_i^h , so that the Radon-Nikodym derivative $d\mathbb{P}/d\mathbb{Q}_i^h$ restricted to \mathcal{F}_{t_j} writes

$$\begin{aligned} \frac{d\mathbb{P}}{d\mathbb{Q}_i^h}|_{\mathcal{F}_{t_j}} &:= L_{i+1,j}^{(h)} \mathbf{1}_{j>i+1} + \mathbf{1}_{j \leq i+1} \quad (= L_{i+1,j}^{(h)} \mathbf{1}_{j>i} + \mathbf{1}_{j \leq i} \text{ because } L_{i+1,i+1}^{(h)} = 1) \\ &= \mathbf{1}_{j>i+1} \exp \left(- \sum_{k=i+1}^{j-1} \left\{ h_k \Delta W_k^{\mathbb{Q}_i^h} + \frac{1}{2} |h_k|^2 \Delta_k \right\} \right) + \mathbf{1}_{j \leq i+1} \end{aligned}$$

for $j \geq i$. Now, we are in a position to represent the MWDP (6) with the above change of probability measure: for $i = N-1, \dots, 0$, set

$$\left. \begin{aligned} Y_i &= \mathbb{E}_{\mathbb{Q}_i^{h^N}, i} \left[\xi L_{i+1,N}^{(h^N)} + \sum_{k=i}^{N-1} f(t_k, Y_{k+1}) L_{i+1,k+1}^{(h^N)} \Delta_k \right], \\ Z_i &= \mathbb{E}_{\mathbb{P}, i} \left[\xi \Theta_N^{(i)} + \sum_{k=i+1}^{N-1} f(t_k, Y_{k+1}) \Theta_k^{(i)} \Delta_k \right], \\ h_i^N &= Z_i/Y_i, \\ \mathbb{E}_{\mathbb{Q}_i^{h^N}, i}[\cdot] &= \mathbb{E}_{\mathbb{Q}_i^{h^N}}[\cdot | \mathcal{F}_{t_i}]. \end{aligned} \right\} \quad (7)$$

The importance sampling DPEs (7) are the natural discrete-time approximation of (5). They are solved recursively backwards in time using the pseudo-algorithm

$$[Y_N(=\xi)] \rightarrow [Z_{N-1} \rightarrow Y_{N-1} \rightarrow h_{N-1}^N \rightarrow \mathbb{Q}_{N-2}^{h^N}] \rightarrow [Z_{N-2} \rightarrow Y_{N-2} \rightarrow h_{N-2}^N \rightarrow \mathbb{Q}_{N-3}^{h^N}] \rightarrow \dots$$

Observe that there is no importance sampling for the Z component of the solution; we will comment on this in Section 2.5.

In what follows, we specialize to a Markovian setting. The standing assumptions are the following.

(H_R) The Brownian motion W and the initializing random variable R are independent.

(H_X)² The drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ of (2) is bounded and the diffusion coefficient $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^q$ is bounded, uniformly η -Hölder continuous in space, $\eta > 0$, uniformly in time. Furthermore, $\sigma\sigma^\top$ is uniformly elliptic, i.e. there exists a constant $C_X \geq 1$ such that for any $t \in [0, T]$, any $x \neq y \in \mathbb{R}^d$, any $\zeta \in \mathbb{R}^q$, we have

$$|b(t, x)| + \frac{|\sigma(t, x) - \sigma(t, y)|}{|x - y|^\eta} \leq C_X, \quad C_X^{-1}|\zeta|^2 \leq \zeta^\top \sigma(t, x) \sigma(t, x)^\top \zeta \leq C_X |\zeta|^2.$$

The time grid $\pi = \{0 := t_0 < \dots < t_i < \dots < t_N := T\}$ is uniform, i.e. $t_i = Ti/N$ and $\Delta_i = T/N$ for all i . The Euler scheme associated to $X^{(R)}$ with time-step Δ_i is denoted by $X_{t_i}^{(N, R)} := X_i^{(R)}$ and defined iteratively by $X_0^{(R)} = R$ and $X_{i+1}^{(R)} = X_i^{(R)} + b_i(X_i^{(R)})\Delta_i + \sigma_i(X_i^{(R)})\Delta W_i$ for $i \geq 0$, where we set $b_i(\cdot) = b(t_i, \cdot)$ and $\sigma_i(\cdot) = \sigma(t_i, \cdot)$.

(H_f) In (6), $f(t_i, y, \omega) = f_i(X_i^{(R)}, y)$ a.s., where $f_i(x, y)$ is a deterministic function, globally Lipschitz continuous in y uniformly in i and x (with a Lipschitz constant L_f), and $|f_i(x, 0)| \leq C_f$ uniformly in i and x ;

(H_ξ) $\xi(\omega) = g(X_N^{(R)})$, where g is a bounded globally Lipschitz continuous function (with Lipschitz constant L_g);

(H_C) $1 \leq Y_i \leq C_y$ and $|Z_i| \leq C_z$ for any i .

(H_Θ) For any i , the Malliavin weights $\Theta^{(i)} := (\Theta_{i+1}^{(i)}, \dots, \Theta_N^{(i)})$ in (6) are squared integrable $(\mathbb{R}^q)^\top$ -valued random variables and there exist measurable functions $\theta_j^{(i)}$ such that

$$\Theta_j^{(i)} = \theta_j^{(i)}(X_i^{(R)}, \dots, X_N^{(R)}, \Delta W_i, \dots, \Delta W_{N-1}).$$

There is a finite constant $C_M \in (0, +\infty)$ such that, for any $0 \leq i < k \leq N$,

$$\mathbb{E} [\Theta_k^{(i)} | X_i] = 0, \quad \mathbb{E} [|\Theta_k^{(i)}|^2 | X_i]^{1/2} \leq \frac{C_M}{\sqrt{t_k - t_i}}.$$

(H_C) is a reinforcement of **(H_a)**-**(H_b)**, which ensures that h^N is bounded. The latter property is frequently used in the subsequent analysis. Observe also that the above assumptions are

²This coincides with assumptions **(UE)** and **(SB)** of [24].

stronger than those for the MWDP scheme with no importance sampling [18] (for the time grid, $\Delta_i = T/N$ and $R_\pi = 1$); this is in the spirit of simplifying the paper.

In the case where X is a Brownian motion, we can take $\Theta_k^{(i)} = \frac{W_{t_k} - W_{t_i}}{t_k - t_i}$ (see [18, Section 1.4]), which obviously satisfies (\mathbf{H}_Θ) . From [28], we know that the continuous time Malliavin weight $\Theta_s^{(t)}$ is a Brownian stochastic integral between t and s with the integrand depending on the SDE. Thus, the assumption (\mathbf{H}_Θ) is an adaptation of this property to a discrete time setting. We allow $\Theta_j^{(i)}$ to depend on the processes between j and N , although this dependence does not appear in explicit examples we are aware of [34], and this dependence is treated in the subsequent analysis.

Remark 2.3. Observe that, under the probability measure \mathbb{Q}^h defined by $d\mathbb{Q}^h|_{\mathcal{F}_{t_i}}/d\mathbb{P}|_{\mathcal{F}_{t_i}} = [L_i^{(h)}]^{-1}$, the \mathbb{P} -marginals $(W_{t_i} : 0 \leq i \leq N)$ coincide in distribution with the \mathbb{Q}^h -marginals $(W_{t_i}^{\mathbb{Q}^h} - \sum_{j=0}^{i-1} h_j^\top \Delta_j : 0 \leq i \leq N)$, where $W^{\mathbb{Q}^h}$ is a \mathbb{Q}^h -Brownian motion. The dynamics of the Euler scheme of (\mathbf{H}_X) are given by

$$X_0^{(R)} = R, \quad X_{i+1}^{(R)} = X_i^{(R)} + \left(b_i(X_i^{(R)}) + \sigma(t_i, X_i^{(R)}) h_i^\top \right) \Delta_i + \sigma_i(X_i^{(R)}) \Delta W_i^{\mathbb{Q}^h}. \quad (8)$$

Under (\mathbf{H}_f) and (\mathbf{H}_ξ) , one can derive the upper bounds in (\mathbf{H}_C) from [18, Corollary 2.6]; in particular, one needs the Lipschitz continuity of the function g to ensure that the bound on $|Z_i|$ is uniform in i . (\mathbf{H}_C) assumes additionally a lower bound on Y that can be obtained as described in Remark 2.2. Moreover, one can easily obtain the following Markovian property, which follows by applying a reverse change of measure in the expression of Y_i and using the results of [18, Section 3.1].

Lemma 2.4. Assume (\mathbf{H}_R) , (\mathbf{H}_X) , (\mathbf{H}_f) , (\mathbf{H}_ξ) , (\mathbf{H}_Θ) . For each i , there exist measurable functions $y_i : \mathbb{R}^d \rightarrow \mathbb{R}$ and $z_i : \mathbb{R}^d \rightarrow (\mathbb{R}^q)^\top$ such that $Y_i = y_i(X_i^{(R)})$ and $Z_i = z_i(X_i^{(R)})$.

Our aim is to approximate the functions y_i and z_i by LSMC with a reduced statistical error on y_i ; the precise scheme is given in Section 2.3 below. Due to Lemma 2.4, y_i and z_i do not depend on the way we initialize the forward SDE using the random variable R . The choice of R gives an additional degree of freedom which we will make use of in the algorithm and analysis. The appropriate choice of R will become clear later in coming Sections 3 and 5 when we will study the algorithm convergence.

W.l.o.g. we assume from now on that y_i and z_i fulfill the inequalities (\mathbf{H}_C) too.

2.3 Importance sampling Monte Carlo scheme

Our aim is to approximate the functions y_i and z_i by LSMC. As inputs, our algorithm will take basis functions as well as independent simulations of random variables. These random variables are defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. On the other hand, the usual stochastic processes and random variables describing the (discrete) BSDE are defined on a probability

space $(\Omega, \mathcal{F}, \mathbb{P})$. Strictly speaking, we work with the product space $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}})$ to describe and analyse the algorithm. However, this rigorous treatment complicates the notation without bringing unexpected features. For simplicity, we avoid $(\omega, \tilde{\omega})$ -notation whenever clear.

Before introducing the Monte Carlo scheme, we introduce the following class of Markov chains. Let $(R^{(0)}, \dots, R^{(N-1)})$ be i.i.d. copies of a random variable R and W be a Brownian motion. We assume that they are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and mutually independent; furthermore let $\mathbf{h} : (\tilde{\omega}, i, x) \in \tilde{\Omega} \times \{0, \dots, N-1\} \times \mathbb{R}^d \rightarrow h_i(\tilde{\omega}, x) \in (\mathbb{R}^q)^\top$ be a bounded stochastic drift function. We will write $\mathbf{X}^{(i)}(\mathbf{h})$ to denote the path

$$\left(X_i^{(i)}(\mathbf{h}) = R^{(i)}, X_{i+1}^{(i)}(\mathbf{h}), \dots, X_N^{(i)}(\mathbf{h}) \right),$$

where $(X_j^{(i)}(\mathbf{h}))_{i \leq j \leq N}$ is (conditionally on $\tilde{\mathcal{F}}$) a Markov chain given by

$$\begin{cases} X_i^{(i)}(\mathbf{h}) &:= R^{(i)}, & X_{i+1}^{(i)}(\mathbf{h}) &:= X_i^{(i)}(\mathbf{h}) + b_i(X_i^{(i)}(\mathbf{h}))\Delta_i + \sigma_i(X_i^{(i)}(\mathbf{h}))\Delta W_i, \\ X_{j+1}^{(i)}(\mathbf{h}) &:= X_j^{(i)}(\mathbf{h}) + \left[b_j(X_j^{(i)}(\mathbf{h})) + \sigma_j(X_j^{(i)}(\mathbf{h}))\mathbf{h}_j^\top(X_j^{(i)}(\mathbf{h})) \right] \Delta_j + \sigma_j(X_j^{(i)}(\mathbf{h}))\Delta W_j, \\ &&& i+1 \leq j \leq N-1; \end{cases} \quad (9)$$

for simplicity, we do not explicit the $\tilde{\omega}$ -dependency. It is clear that the Markov chain $\{X_j^{(R)}\}_{j \geq 0}$ of Assumption $(\mathbf{H}_\mathbf{X})$ is given by $\{X_j^{(0)}(0)\}_{j \geq 0}$. Now, for every $j \geq i$, we recall the functions $y_i(\cdot)$ and $z_i(\cdot)$ from Lemma 2.4 in order to introduce

$$h_j(\cdot) := z_j(\cdot)/y_j(\cdot),$$

$$L_{i+1,j}^h(\mathbf{x}, \mathbf{w}) := \exp \left(- \sum_{k=i+1}^{j-1} \left\{ h_k(x_k)w_k + \frac{1}{2}|h_k(x_k)|^2 \Delta_k \right\} \right),$$

where $\mathbf{x} = (x_i, \dots, x_N) \in (\mathbb{R}^d)^{N-i+1}$ and $\mathbf{w} = (w_i, \dots, w_{N-1}) \in (\mathbb{R}^q)^{N-i}$. Set

$$\begin{aligned} S_{Y,i}(\mathbf{x}, \mathbf{w}) &:= g(x_N) L_{i+1,N}^h(\mathbf{x}, \mathbf{w}) + \sum_{k=i}^{N-1} f_k(x_k, y_{k+1}(x_{k+1})) L_{i+1,k+1}^h(\mathbf{x}, \mathbf{w}) \Delta_k, \\ S_{Z,i}(\mathbf{x}, \mathbf{w}) &:= g(x_N) \theta_N^{(i)}(\mathbf{x}, \mathbf{w}) + \sum_{k=i+1}^{N-1} f_k(x_k, y_{k+1}(x_{k+1})) \theta_k^{(i)}(\mathbf{x}, \mathbf{w}) \Delta_k. \end{aligned} \quad (10)$$

The following lemma has exactly the same proof as Lemma 2.4, and makes use of Remark 2.3 and (8).

Lemma 2.5. *Assume $(\mathbf{H}_\mathbf{R})$, $(\mathbf{H}_\mathbf{X})$, $(\mathbf{H}_\mathbf{f})$, (\mathbf{H}_ξ) , (\mathbf{H}_Θ) . For each i , almost surely it holds that*

$$y_i(R^{(i)}) = \mathbb{E}_\mathbb{P} \left[S_{Y,i}(\mathbf{X}^{(i)}(h), \Delta \mathbf{W}^{(i)}) | R^{(i)} \right], \quad z_i(R^{(i)}) = \mathbb{E}_\mathbb{P} \left[S_{Z,i}(\mathbf{X}^{(i)}(0), \Delta \mathbf{W}^{(i)}) \mid R^{(i)} \right],$$

where $\Delta \mathbf{W}^{(i)} := (\Delta W_i, \dots, \Delta W_{N-1})$.

In the remainder of this section, we approximate the value functions $y_i(\cdot)$ and $z_i(\cdot)$ from Lemma 2.4 with numerical counterparts $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$, respectively, using a fully implementable LSMC algorithm. We call this scheme the Importance Sampling Malliavin Weights LSMC scheme (ISMWDP).

Let $h_i^{(M)}(x) := z_i^{(M)}(x)/y_i^{(M)}(x)$, and set, for $\mathbf{x} = (x_i, \dots, x_N) \in (\mathbb{R}^d)^{N-i+1}$ and $\mathbf{w} = (w_i, \dots, w_{N-1}) \in (\mathbb{R}^q)^{N-i}$,

$$\begin{aligned} L_{i+1,j}^{h(M)}(\mathbf{x}, \mathbf{w}) &:= \exp \left(- \sum_{k=i+1}^{j-1} \left\{ h_k^{(M)}(x_k) w_k + \frac{1}{2} |h_k^{(M)}(x_k)|^2 \Delta_k \right\} \right), \\ S_{Y,i}^{(M)}(\mathbf{x}, \mathbf{w}) &:= g(x_N) L_{i+1,N}^{h(M)}(\mathbf{x}, \mathbf{w}) + \sum_{k=i}^{N-1} f_k(x_k, y_{k+1}^{(M)}(x_{k+1})) L_{i+1,k+1}^{h(M)}(\mathbf{x}, \mathbf{w}) \Delta_k, \\ S_{Z,i}^{(M)}(\mathbf{x}, \mathbf{w}) &:= g(x_N) \theta_N^{(i)}(\mathbf{x}, \mathbf{w}) + \sum_{k=i+1}^{N-1} f_k(x_k, y_{k+1}^{(M)}(x_{k+1})) \theta_k^{(i)}(\mathbf{x}, \mathbf{w}) \Delta_k. \end{aligned}$$

Define also the threshold functions

$$\mathcal{T}_{C_y}(y) := 1 \vee y \wedge C_y, \quad \mathcal{T}_{C_z}(z) := \left(-C_z \vee z_1 \wedge C_z, \dots, -C_z \vee z_q \wedge C_z \right)$$

for $y \in \mathbb{R}$ and $z \in (\mathbb{R}^q)^\top$, where the constants C_y and C_z are defined in (\mathbf{H}_C) .

We now come to the ISMWDP algorithm.

Algorithm 2.1. Set $y_N^{(M)}(\cdot) = g(\cdot)$. Starting from $i = N - 1$ and working backwards to $i = 0$, let

- $K^{(Y,i)} \in \mathbb{N}^*$ and $\{H_1^{(Y,i)}, \dots, H_{K^{(Y,i)}}^{(Y,i)}\}$ be disjoint subsets of \mathbb{R}^d , and $D^{(Y,i)} := \bigcup_k H_k^{(Y,i)}$. On each $H_k^{(Y,i)}$, y_i will be approximated by a constant (a.k.a. partitioning regression estimate). As shown later in Section 4, the use of such specific basis functions allows to retrieve the variance reduction effect in the error estimates for the regression problems.
- $\mathcal{K}_{Z,i} = \text{span}\{p_1^{(i)}, \dots, p_{K^{(Z,i)}}^{(i)} : \mathbb{E}[|p_j^{(i)}(R)|^2] < \infty \forall j\}$ for functions $p_k^{(i)} : \mathbb{R}^d \rightarrow \mathbb{R}$, $k \in \{1, \dots, K^{(Z,i)}\}$, where $K^{(Z,i)} \in \mathbb{N}^*$. This linear space will serve for approximating z_i . We allow full generality for the basis functions.
- $M_i \in \mathbb{N}^*$ a number of simulations at time t_i , with $M_i \geq \max(K^{(Y,i)}, K^{(Z,i)})$ (to avoid having an under-determined system).
- $\mathcal{C}_i := \{(R^{(i,1)}, \Delta \mathbf{W}^{(i,1)}), \dots, (R^{(i,M_i)}, \Delta \mathbf{W}^{(i,M_i)})\}$ be a collection of i.i.d. copies of random variables $R^{(i)}$ and the Brownian increments $\Delta \mathbf{W}^{(i)}$; we term \mathcal{C}_i to be the cloud of simulations at time t_i .
- The clouds of simulations $\{\mathcal{C}_0, \dots, \mathcal{C}_{N-1}\}$ are mutually independent, i.e. independently simulated.

- The cloud \mathcal{C}_i is used to construct i.i.d. copies of the Markov chains $\mathbf{X}^{(i)}(h^{(M)})$ and $\mathbf{X}^{(i)}(0)$, and Malliavin weights $\Theta^{(i)}$. The trajectories of the simulations are termed $\{\mathbf{X}^{(i,m)}(h^{(M)}) : m = 1, \dots, M_i\}$ and $\{\mathbf{X}^{(i,m)}(0) : m = 1, \dots, M_i\}$.

Set the sample dependent functions $y_i^{(M)}(\cdot)$, $z_i^{(M)}(\cdot)$ and $h_i^{(M)}(\cdot)$ recursively as follows.

a) Approximation of z_i . Define

$$\psi_{Z,i}^{(M)}(\cdot) := \arg \inf_{\phi \in \mathcal{K}_{Z,i}} \frac{1}{M} \sum_{m=1}^M \left(S_{Z,i}^{(M)}(\mathbf{X}^{(i,m)}(0), \Delta \mathbf{W}^{(i,m)}) - \phi(R^{(i,m)}) \right)^2,$$

and we set $z_i^{(M)}(\cdot) := \mathcal{T}_{C_z}(\psi_{Z,i}^{(M)}(\cdot))$.

b) Approximation of y_i . For every $k \in \{1, \dots, K^{(Y,i)}\}$, define the set

$$A_k^{(Y,i)} := \left\{ m \in \{1, \dots, M_i\} : R^{(i,m)} \in H_k^{(Y,i)} \right\}.$$

Let $\psi_{Y,i}^{(M)}(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ be a piecewise constant function defined as follows:

1. if $A_k^{(Y,i)} = \emptyset$, set $\psi_{Y,i}^{(M)}(\cdot)|_{H_k^{(Y,i)}} := 1$;
2. otherwise, set

$$\psi_{Y,i}^{(M)}(\cdot)|_{H_k^{(Y,i)}} := \frac{1}{\#(A_k^{(Y,i)})} \sum_{m \in A_k^{(Y,i)}} S_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \Delta \mathbf{W}^{(i,m)}). \quad (11)$$

Last, set $\psi_{Y,i}^{(M)}(\cdot)|_{[D^{(Y,i)}]^c} := 1$ and $y_i^{(M)}(\cdot) := \mathcal{T}_{C_y}(\psi_{Y,i}^{(M)}(\cdot))$.

c) Approximation of h_i . Having calculated $z_i^{(M)}(\cdot)$ and $y_i^{(M)}(\cdot)$, define $h_i^{(M)}(\cdot) = z_i^{(M)}(\cdot)/y_i^{(M)}(\cdot)$.

The convergence analysis of the above algorithm ISMWDP is postponed to Section 5, after we establish some essential mathematical techniques for this purpose.

Remarks.

1. The Euler schemes $\mathbf{X}^{(i)}(h^{(M)})$ and $\mathbf{X}^{(i)}(0)$ are re-simulated at every time-point t_i , which is necessary to account for the updated IS drift. At a first sight, this re-simulation step seems computationally expensive. In fact, re-simulation serves the additional purpose of reducing memory consumption, since the memory allocation of each simulation can be immediately removed once its contribution to the regression has been made. This is explained in the introduction of [19]. This is important in practice, because we know from [19, 18] that the memory is the critical point for LSMC-based schemes in large dimension.

2. We are now finally in a position to motivate why each Markov chain $\mathbf{X}^{(i)}(h^{(M)})$ is started from an arbitrary random variable R at time i rather than from a fixed point at time zero. Inspecting (9) with $\mathbf{h} = h^{(M)}$, we see that we need the functions $y_j^{(M)}(\cdot)$ and $z_j^{(M)}(\cdot)$ in order to generate the trajectories of the Markov chain $\mathbf{X}^{(i)}(h^{(M)})$. This is implementable if we were to simulate starting from time point i . On the other hand, since at time point i the functions $y_j^{(M)}(\cdot)$ and $z_j^{(M)}(\cdot)$ for $j \leq i$ are unknown, it would not be possible to simulate trajectories of $\mathbf{X}^{(i)}(h^{(M)})$ starting from time 0, so the scheme would not be implementable.
3. The basis functions $\mathcal{K}_{Y,i}$ and $\mathcal{K}_{Z,i}$ are chosen according to the smoothness of the functions $y(t, \cdot)$ and $z(t, \cdot)$, which are usually known. We refer the reader to [19][18] for detailed discussions.
4. In order to obtain explicit error estimates, we cannot in fact take an arbitrary distribution for R . We need to complement the assumption (\mathbf{H}_R) with assumption (\mathbf{H}_ρ) in Section 3 below. We briefly explain why. Let $i_1, i_2 \in \{0, \dots, N-1\}$ be unequal $i_1 \neq i_2$. It is clear from the Euler scheme (9) that the random variables $X_j^{(i_1)}(\mathbf{h})$ and $X_j^{(i_2)}(\mathbf{h})$ will have **unequal** distributions whenever $j > \min(i_1, i_2) + 1$. This implies that the \mathbf{L}_2 -norms $\|\phi\|_{i,j} := (\mathbb{E}[\phi(X_j^{(i)}(\mathbf{h}))^2])^{1/2}$ do not correspond at j : $\|\phi\|_{i_1,j} \neq \|\phi\|_{i_2,j}$. If these norms were to coincide, the usual approach [18] is to make use of Gronwall's inequality to treat the propagation of error terms due to dynamical programming. Assumption (\mathbf{H}_ρ) in Section 3 allows to prove that all the quantities $\|\phi\|_{i,j}$ are equivalent up to a constant, uniformly in ϕ , and thus we recover the ability to use Gronwall's inequality. We recall that whatever the choice R is, it does not affect the definitions of y_i and z_i . Note that in usual LSMC algorithms [19, 18, Proposition 2.4], where the Markov chains are initialized starting from a fixed point at time zero uniformly for every i , one does not face this problem.
5. In (10), we could replace $g(x_N)$ by $g(x_N) - g(x_i)$. This would not change the equality in the conditional expectations because the Malliavin weights are conditionally centered. However this would additionally reduce the variance (see [1]). In the subsequent error analysis (see the proof of Theorem 2.6), a careful inspection shows that this would remove the factor $1/(T - t_i)$ in the definition of $\mathcal{E}^{(Z,i)}$ under the condition, in addition to (\mathbf{H}_Θ) , that Malliavin weights have finite fourth moments satisfying $\mathbb{E}[|\Theta_j^{(i)}|^4]^{1/4} \leq C_M(t_j - t_i)^{-1/2}$.

2.4 Main error result

We now estimate the accuracy of the functions $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$ as approximates for the functions $y_i(\cdot)$ and $z_i(\cdot)$ of Lemma 2.4 in terms of the choice basis functions and the number of simulations. It is well known that $y_i(\cdot)$ and $z_i(\cdot)$ converge to the solution of the continuous

time BSDE $y(t, \cdot)$ and $z(t, \cdot)$ as the time grid becomes infinitely fine [34], so the below results serve to show the convergence to the continuous time BSDE. Our main error estimates on the ISMWDP algorithm are the following.

Theorem 2.6. *Assume (\mathbf{H}_X) , (\mathbf{H}_f) , (\mathbf{H}_ξ) , (\mathbf{H}_C) , (\mathbf{H}_Θ) , (\mathbf{H}_ρ) ³. For each i , define the function*

$$\hat{S}_{Y,i}^{(M)}(\mathbf{x}, \mathbf{w}) := g(x_N) L_{i+1,N}^{h(M)}(\mathbf{x}, \mathbf{w}) + \sum_{k=i}^{N-1} f_k(x_k, y_{k+1}(x_{k+1})) L_{i+1,k+1}^{h(M)}(\mathbf{x}, \mathbf{w}) \Delta_k$$

and set

$$\text{osc}_k^{(y_i)} := \sup_{x, x' \in H_k^{(Y,i)}} |y_i(x) - y_i(x')|, \quad p_k^{(Y,i)} := \mathbb{P}\left(R^{(i)} \in H_k^{(Y,i)}\right),$$

$$\sigma_{Y,i,M}^2 := \mathbb{E} \left[\text{ess sup}_{x \in \mathbb{R}^d} \mathbb{V}\text{ar} \left[\hat{S}_{Y,i}^{(M)}(\mathbf{X}^{(i)}(h^{(M)}), \Delta \mathbf{W}^{(i)}) | R^{(i)} = x, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1} \right] \right],$$

$$\mathcal{E}^{(Y,i)} := \sum_{k=1}^{K^{(Y,i)}} [\text{osc}_k^{(y_i)}]^2 p_k^{(Y,i)} + \sigma_{Y,i,M}^2 \frac{K^{(Y,i)}}{M_i} + C_y^2 \sum_{k=1}^{K^{(Y,i)}} p_k^{(Y,i)} \exp\left(-M_i p_k^{(Y,i)}\right) + C_y^2 \mathbb{P}(R^{(i)} \notin D^{(Y,i)}),$$

$$\mathcal{E}^{(Z,i)} := \inf_{\varphi \in \mathcal{K}_{Z,i}} \|z_i(\cdot) - \varphi(\cdot)\|_\rho^2 + C_z^2 \frac{K^{(Z,i)} q \log(3M_i)}{M_i} + \frac{C_{(45)}^2}{T - t_i} \frac{K^{(Z,i)}}{M_i}.$$

for constant $C_{(45)}$ to be defined in (45) below. There exists a constant $C_{(12)}$ (resp. $C_{(13)}$) depending only on T , L_f , C_z , $\bar{c}_{(16)}$, $C_{(29)}$ (resp. on T , L_f , C_z , C_M , $\bar{c}_{(16)}$, $C_{(29)}$) such that, for each $i \in \{0, \dots, N-1\}$

$$\mathbb{E} \left[\left\| y_i(\cdot) - y_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq C_{(12)} \left((\mathcal{E}^{(Y,i)})^{1/2} + \sum_{k=i+1}^{N-1} (\mathcal{E}^{(Y,k)})^{1/2} \Delta_k \right), \quad (12)$$

$$\mathbb{E} \left[\left\| z_i(\cdot) - z_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq C_{(13)} \left((\mathcal{E}^{(Z,i)})^{1/2} + \sum_{k=i+1}^{N-1} \frac{(\mathcal{E}^{(Y,k+1)})^{1/2}}{\sqrt{t_k - t_i}} \Delta_k \right). \quad (13)$$

We prove this theorem in Section 5, after establishing some necessary techniques in the next two sections. As in [18, Section 3.5], the above error estimates are sufficient to establish the convergence of the ISMWDP scheme by appropriately choosing the basis functions and the number of simulations. The outer boundary of $D^{(Y,i)}$ is determined by the distribution of R : we set it so that $\mathbb{P}(R^{(i)} \notin D^{(Y,i)})$ is sufficiently small, depending on accuracy requirements. We expect that the importance sampling algorithm will converge faster w.r.t. the simulation effort compared to basic LSMC scheme. The improvement can be captured through the term

³See Section 3 for (\mathbf{H}_ρ) .

$\sigma_{Y,i,M}^2$, which is expected to be small due to Proposition 2.1; the exponential sums in $\mathcal{E}^{(Y,i)}$ are negligible for sufficiently large M_i . $\sigma_{Y,i,M}^2$ should be compared with $\mathcal{E}_{\text{Dep},i}^Y$ in [18, Proposition 3.9]; the later is constant, regardless of the time grid or the basis functions, and number of simulations, whereas the former is expected to converge to zero as the time grid and basis functions become finer and the number of simulations becomes large. In Section 6, we perform a number of numerical experiments. We plot histograms from which both bias term $\text{osc}_k^{(y_i)}$ and the variance term $\sigma_{Y,i,M}^2$ can be determined for specific examples. Actually, the finiteness of $\sigma_{Y,i,M}^2$ (with uniform bound w.r.t. i) easily follows from the boundedness of the IS-drift, the boundedness of $y(\cdot)$, and of the boundedness/Lipschitz continuity of f_k . Observe that the function $\hat{S}_{Y,i}^{(M)}(\mathbf{x}, \mathbf{w})$ is constructed with the true function $y_{k+1}(\cdot)$ in the driver f_k , and with the empirical drift $h_k^{(M)}(\cdot) \approx h_k(\cdot)$ in the Radon-Nikodym derivative function $L_{i+1,k+1}^{h^{(M)}}(\mathbf{x}, \mathbf{w})$. Although it is delicate to precisely quantify the variance reduction, the statistical error of the ISMWDP is presumably much smaller compared to a scheme without IS. This is well illustrated in Figures 1 and 2 in Section 6.

2.5 Digression on the driver not depending on z

In Section 2.1, the optimal importance sampling for computing the conditional expectation of \mathcal{Y} is obtained using the drift \mathcal{Z}/\mathcal{Y} . It is not clear how to efficiently transfer these arguments to \mathcal{Z} . Below, we suggest two possible approaches. Both approaches have some potential, but suffer from technical difficulties which we do not know how to solve at present. We explore these difficulties to encourage future research on this topic.

1. First, from the representation of Z in (6) using the Malliavin integration-by-parts formula, we obtain the l -th component of \mathcal{Z}_{t_i} (or its approximation Z_i) as the conditional expectation of $\xi_{l,i} := \xi_{l,T}^{(t_i)} + \int_{t_i}^T \Theta_{l,s}^{(t_i)} f(s, \mathcal{Y}_s, \mathcal{Z}_s) ds$. Therefore, it can be interpreted as a new BSDE problem on the interval $[t_i, T]$ with zero driver and terminal condition $\xi_{l,i}$. Denote its solution by $(\mathcal{Y}^{\xi_{l,i}}, \mathcal{Z}^{\xi_{l,i}})$. Using the techniques of Proposition 2.1 we obtain that the optimal drift for changing the measure for the evaluation of \mathcal{Z}_{l,t_i} is (formally) $h^{\xi_{l,i}} := \mathcal{Z}^{\xi_{l,i}}/\mathcal{Y}^{\xi_{l,i}}$. This leads to significant difficulties: first from the theoretical point of view, there is no clear set of checkable assumptions ensuring we can reduce to (\mathbf{H}_a) - (\mathbf{H}_b) for the new BSDE. Second, from the numerical point of view, one must solve a BSDE $(\mathcal{Y}^{\xi_{l,i}}, \mathcal{Z}^{\xi_{l,i}})$ for every time-point t_i in order to obtain the optimal probability measure. Computationally, this is extremely expensive and it seems a priori that there is no way such an algorithm may be efficient.
2. Second, instead of the representation of \mathcal{Z}_{t_i} or Z_i using integration by parts formula, we could take advantage of the BSDE-type equation satisfied by $(\mathcal{Z}_t)_t$ (see [21] for a recent account on the subject). However, these equations involve “the \mathcal{Z} of the \mathcal{Z} ”, i.e. Gamma processes. We must add DPEs to (7) to approximate the Gamma (like in

2BSDE [13]). However, a complete error analysis of these DPEs in the context of LSMC algorithms seems especially difficult. Therefore, this approach is beyond the scope of this work.

Finally, if $f(t, y, z)$ depends on z and if the Monte-Carlo estimation of Z in (7) is made without appropriate variance reduction (suited to Z specifically), we would obtain a propagation of “lack of variance reduction” on the Y component due to the Z component through f . Therefore, we would lose the variance reduction on Y . Thus, to keep track of the benefit of importance sampling for Y , it seems necessary to consider a driver independent of z .

3 Stability of L_2 -norm under USES

In the following section, we develop some technical results for to the choice of the initializing random variable R on the process $\mathbf{X}^{(i)}(\mathfrak{h})$. These results are essential for the proof of Theorem 2.6, and as such will determine how we choose R in practice.

Let ρ be a probability density w.r.t. the Lebesgue measure on \mathbb{R}^d , and R be a random variable with such a density. Set

$$\|\varphi\|_\rho := \left(\int_{\mathbb{R}^d} \varphi^2(x) \rho(x) dx \right)^{1/2} = (\mathbb{E} [\varphi^2(R)])^{1/2}$$

for any measurable function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ in $L_2(\mathbb{R}^d, \rho)$. We now introduce the USES assumption.

(H $_\rho$) ρ is a continuous density and there is a positive continuous function $C_\rho : \mathbb{R} \rightarrow [1, +\infty)$ such that, for any $\Lambda \geq 0$, any $\lambda \in [0, \Lambda]$ and any $y \in \mathbb{R}^d$, we have

$$\frac{\rho(y)}{C_\rho(\Lambda)} \leq \int_{\mathbb{R}^d} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{d/2}} dz \leq C_\rho(\Lambda) \rho(y). \quad (14)$$

We present tractable examples of ρ in Proposition 3.3, and state how to simulate the corresponding random variable R ; all have sub-exponential tails. Observe that (14) implies that ρ must be strictly positive. The acronym USES stands for Uniform Sub-Exponential Sandwiching and it can be summerized shortly as follows: by initializing a Euler scheme with a density ρ , the marginal density of the process remains equivalent to ρ (up to constant which is uniform locally w.r.t. time). This stability property is stated as follows.

Proposition 3.1. *Assume **(H $_\mathbf{X}$)**. Let $\mathfrak{h} := (\mathfrak{h}_0, \dots, \mathfrak{h}_{N-1})$ be a vector of functions, where $\mathfrak{h}_k : \mathbb{R}^d \rightarrow (\mathbb{R}^q)^\top$ are bounded and measurable. Let $i \in \{0, \dots, N-1\}$, and $R^{(i)}$ be a random variable satisfying **(H $_\rho$)** which is independent of the Brownian motion W . Let $X^{(i)}(\mathfrak{h})$ be the Markov chain*

$$X_i^{(i)}(\mathfrak{h}) := R^{(i)}, \quad (15)$$

$$X_{j+1}^{(i)}(\mathbf{h}) := X_j^{(i)}(\mathbf{h}) + \left[b_j(X_j^{(i)}(\mathbf{h})) + \sigma_j(X_j^{(i)}(\mathbf{h})) \mathbf{h}_j^\top (X_j^{(i)}(\mathbf{h})) \right] \Delta_j + \sigma_j(X_j^{(i)}(\mathbf{h})) \Delta W_j,$$

for $i \leq j \leq N-1$. Then, there exist finite positive constants $\underline{c}_{(16)}$ and $\bar{c}_{(16)}$ (depending only on $d, q, T, \|\mathbf{h}\|_\infty, C_X, \rho$, but not on i) such that, for any $j \in \{i, \dots, N\}$ and any square integrable function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$, we have

$$\underline{c}_{(16)} \|\varphi\|_\rho \leq \left(\mathbb{E} \left[\varphi^2(X_j^{(i)}(\mathbf{h})) \right] \right)^{1/2} \leq \bar{c}_{(16)} \|\varphi\|_\rho. \quad (16)$$

Similar equivalence-of-norm results are established in [2, Proposition 5.1] but they have been derived for time-homogenous diffusion processes (and not Euler schemes) with smooth coefficients. For an extension to time-inhomogeneous diffusion process with smooth in space coefficients, see [14, Proposition 3.8]. Moreover, [14, Proposition 3.9] partially extends this to the Euler scheme, still with smooth coefficients. Our contribution is to consider drift coefficients b and \mathbf{h} without smoothness condition in space and time, and only Hölder continuity in space for the diffusion coefficient σ . The main application of this is to piecewise-continuous \mathbf{h} (see (9)), as we encounter in the LSMC scheme (Algorithm 2.1).

The reader will easily check from the proof below that the above equivalence of \mathbf{L}_2 -norms immediately extends to \mathbf{L}_p -norms, $p \geq 1$; however, in this work, only the case $p = 2$ is used.

Proof of Proposition 3.1. Let $0 \leq i < j \leq N$ and $x \in \mathbb{R}^d$. Denote by $p_{i,j}^x(y)$ the density at $y \in \mathbb{R}^d$ of the random variable X_j^x defined iteratively by

$$X_i^x := x, \quad X_{j+1}^x := X_j^x + \left[b_j(X_j^x) + \sigma_j(X_j^x) \mathbf{h}_j^\top (X_j^x) \right] \Delta_j + \sigma_j(X_j^x) \Delta W_j \text{ for } j \geq i.$$

It is easy to check that, under the ellipticity assumption $(\mathbf{H}_\mathbf{x})$, this density exists; actually it can be written as a convolution of Gaussian densities. In view of (15), observe that $X_j^{R^{(i)}} := X_j^{(i)}(\mathbf{h})$. The following lemma, proved in [24], provides upper and lower bounds on $p_{i,j}^x(y)$ using a Gaussian density (i.e. Aronson-like estimates).

Lemma 3.2 ([24, Theorem 2.1]). *Under the assumptions and notation of Proposition 3.1, there exists a finite constant $C_{(17)} \geq 1$ (depending only on d, q, T and C_X) such that*

$$\frac{1}{C_{(17)}} \frac{e^{-C_{(17)} \frac{|y-x|^2}{2(t_j-t_i)}}}{(2\pi(t_j-t_i))^{d/2}} \leq p_{i,j}^x(y) \leq C_{(17)} \frac{e^{-\frac{|y-x|^2}{2C_{(17)}(t_j-t_i)}}}{(2\pi(t_j-t_i))^{d/2}}, \quad (17)$$

for any $0 \leq i < j \leq N$ and any $x, y \in \mathbb{R}^d$.

The upper bound of (16) can now be proved as follows:

$$\mathbb{E} \left[\varphi^2(X_j^{(i)}(\mathbf{h})) \right] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x) p_{i,j}^x(y) \varphi^2(y) dx dy$$

$$\begin{aligned} &\leq C_{(17)}^{1+d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho\left(y + z\sqrt{C_{(17)}(t_j - t_i)}\right) \frac{e^{-|z|^2/2}}{(2\pi)^{d/2}} \varphi^2(y) dz dy \\ &\leq C_{(17)}^{1+d/2} C_\rho(C_{(17)}T) \int_{\mathbb{R}^d} \rho(y) \varphi^2(y) dy. \end{aligned}$$

The lower inequality is proved similarly. \square

To conclude this section, we provide a list of distributions satisfying the USES property (\mathbf{H}_ρ) . The proof is postponed to Appendix A.

Proposition 3.3. *The following densities/distributions satisfy Assumption (\mathbf{H}_ρ) ; in addition their coordinates (R_1, \dots, R_d) are i.i.d. and each coordinate can be simply sampled using a Rademacher r.v. ε (taking ± 1 with equal probability) and a $[0, 1]$ -uniformly distributed r.v. U , both being independent.*

(a) Laplace distribution: For $\mu > 0$ set

$$\rho(x) := \prod_{i=1}^d \left[(\mu/2) e^{-\mu|x_i|} \right]. \quad (18)$$

Each coordinate can be sampled as $\varepsilon \ln(U)/\mu$.

(b) Pareto-type distribution: For $\mu > 0$ and $k > 0$, set

$$\rho(x) := \prod_{i=1}^d \left[(\mu k/2) (1 + \mu|x_i|)^{-k-1} \right]. \quad (19)$$

Each coordinate can be sampled as $\varepsilon(U^{-1/k} - 1)/\mu$.

(c) Twisted Exponential-type distribution: For $\mu > 0$ and $\alpha > 2$ set

$$\rho(x) := \prod_{i=1}^d \left[(\mu e/\alpha) e^{-(1+\mu|x_i|)^{2/\alpha}} (1 + \mu|x_i|)^{2/\alpha-1} \right]. \quad (20)$$

Each coordinate can be sampled as $\varepsilon[(1 - \ln(U))^{\alpha/2} - 1]/\mu$.

4 Regression on piecewise constant basis functions and non-parametric estimates

In this section, we develop a fully explicit, nonparametric error analysis for LSMC scheme on piecewise constant functions for a single-period problem. These results are not specific to BSDEs, but are an essential stepping stone to obtaining the error estimates in Theorem 2.6.

As we explain below, typical error estimates in nonparametric statistics do not permit us to capture the variance reduction effect of our importance sampling algorithm, hence we may not use these classical results in our analysis.

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(Y, X) : \Omega \rightarrow \mathbb{R} \times \mathbb{R}^d$ be random variables satisfying $\mathbb{E}[|Y|^2] < +\infty$. Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. copies of these random variables, with $n > 1$. The pair (Y, X) is termed the response/observation, and the ensemble of the i.i.d. copies is termed the *sample* from now on. We aim at estimating the regression function $m(x) = \mathbb{E}[Y | X = x]$ using the sample. The function m is generally unknown. Nonetheless, we assume that $|m(x)| \leq L$ for a known constant $L > 0$.

We denote by ν the joint law of (Y, X) and let ν_n be the empirical measure associated to the sample. Whenever there is no conflict, ν (resp. ν_n) will stand for (by slight abuse of notation) the law of X (resp. the empirical measure associated to the data (X_1, \dots, X_n)).

For $K \in \mathbb{N}^*$ disjoint sets H_1, \dots, H_K in \mathbb{R}^d , let $\mathcal{K} := \text{span}\{\mathbf{1}_{H_1}, \dots, \mathbf{1}_{H_K}\}$ be the linear space of functions that are piecewise constants on each H_k . We estimate the function m using sample and the class of functions \mathcal{K} on the domain $D := \bigcup_{i=1}^K H_i \subset \mathbb{R}^d$ with least-squares regression and truncation:

$$m(\cdot) \approx m_n(\cdot) := \mathcal{T}_L \left[\arg \min_{\varphi \in \mathcal{K}} \int_{\mathbb{R} \times \mathbb{R}^d} |y - \varphi(x)|^2 \nu_n(dy, dx) \right]$$

where $y \in \mathbb{R} \mapsto \mathcal{T}_L[y] = -L \vee y \wedge L$ is the truncation operator⁴. Observe that $m_n(\cdot)$ is *not* the optimal approximation of $m(\cdot)$ in $\mathbf{L}_2(\nu_n)$. This role is played by the function $m_n^*(\cdot)$ defined below:

$$m(\cdot) \approx m_n^*(\cdot) := \arg \min_{\varphi \in \mathcal{K}} \int_{\mathbb{R}^d} |m(x) - \varphi(x)|^2 \nu_n(dx).$$

We emphasize that functions m_n and m_n^* are *piecewise constant* on the sets H_k . Thanks to the orthogonal structure of the class \mathcal{K} , the functions m_n and m_n^* are available in closed form [20, Ch. 4]: on each set H_k , $k \in \{1, \dots, K\}$, the approximating functions are defined by

$$m_n(\cdot)|_{H_k} = \mathcal{T}_L \left[\frac{\int_{\mathbb{R} \times \mathbb{R}^d} y \mathbf{1}_{H_k}(z) \nu_n(dy, dz)}{\nu_n(H_k)} \right], \quad m_n^*(\cdot)|_{H_k} = \frac{\int_{\mathbb{R}^d} m(z) \mathbf{1}_{H_k}(z) \nu_n(dz)}{\nu_n(H_k)} \quad (21)$$

with the convention $0/0 = 0$. In particular, if $\nu_n(H_k) = 0$ (no data in H_k), we set $m_n(\cdot)|_{H_k} = m_n^*(\cdot)|_{H_k} = 0$. This is consistent with the definition of the piecewise constant approximation of y_i given by (11).

The error of the LSMC scheme is given by the expected *risk*

$$R(m_n) := \mathbb{E} \left[\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu(dx) \right]. \quad (22)$$

⁴The truncation operator $\mathcal{T}_L[\cdot]$ serves to correct the least-squares estimate in case it exceeds the a priori known bound. It is a simple improvement which we make use of in proofs, and which yields visible improvements in practice.

The inner integral is taken with respect to the true law ν rather than the empirical law ν_n . One can define the empirical version of (22), i.e.

$$R_n(m_n) := \mathbb{E} \left[\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu_n(dx) \right]. \quad (23)$$

For various reasons, it is important to have switching estimates between $R(m_n)$ and $R_n(m_n)$, i.e. to estimate $R(m_n)$ in terms of $R_n(m_n)$ up to errors, and vice-versa. This is usually achieved using concentration-of-measure inequalities. Let us recall the classical technique to estimate the error. First, estimates on the empirical risk (23) are known [20, Theorem 11.1], and usually take the form

$$R_n(m_n) \leq \text{const} \times \left\{ \sum_{k=1}^K [\text{osc}_k^{(m)}]^2 \nu(H_k) + L^2 \nu(D^c) + K \frac{\sup_{x \in \mathbb{R}^d} \text{Var}(Y | X = x)}{n} \right\} \quad (24)$$

where $\text{osc}_k^{(m)}$ is the oscillation of m on H_k , i.e.

$$\text{osc}_k^{(m)} := \sup_{x, y \in H_k} |m(x) - m(y)|. \quad (25)$$

Then, to complete the estimate, one addresses the difference

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu(dx) - 2 \int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu_n(dx) \right)_+ \right] \\ & \leq \mathbb{E} \left[\sup_{\varphi \in \mathcal{K}, |\varphi|_\infty \leq L} \left(\int_{\mathbb{R}^d} |m(x) - \varphi(x)|^2 \nu(dx) - 2 \int_{\mathbb{R}^d} |m(x) - \varphi(x)|^2 \nu_n(dx) \right)_+ \right] \\ & \leq \frac{2028(K+1) \log(3n) L^2}{n}, \end{aligned} \quad (26)$$

where L is the uniform bound on m . These results are obtained using concentration-of-measure inequalities [19, Propositions 4.9 and 4.10], which force the L^2 to appear by the use of Hoeffding-type inequalities. In the above, we can interchange the roles of ν and ν_n and the upper bound (26) keeps the same form, see the arguments in [19]. The main point to observe is that the estimate from concentration-of-measure (26) is less tight with respect to n than the estimate (24) on $R_n(m_n)$, because the conditional variance $\sup_{x \in \mathbb{R}^d} \text{Var}(Y | X = x)$ may be substantially smaller than L^2 . This is particularly true in the context of variance reduction algorithms, such as the importance sampling scheme presented in Section 2.3, where the aim is to minimize the conditional variance term. Therefore, if one were to use usual concentration-of-measure results like (26), the impact of variance reduction would be lost.

The main result of this section is to demonstrate that, thanks to the structure of the approximation space \mathcal{K} , one can obtain switching estimates on the risks for which estimate on the left-hand side of (26) is much smaller than $L^2 \frac{K \log(n)}{n}$, and therefore for which the impact of variance reduction is not lost.

Theorem 4.1. Assume that m is bounded by $L > 0$. For each $k \in \{1, \dots, K\}$, define $\text{osc}_k^{(m)}$ as in (25). Define also the upper bound $\sigma^2 := \sup_{x \in \mathbb{R}^d} \text{Var}(Y \mid X = x)$. Then

$$R(m_n) \leq 8R_n(m_n) + 10 \sum_{k=1}^K [\text{osc}_k^{(m)}]^2 \nu(H_k) + \frac{8\sigma^2}{n} \sum_{k=1}^K \exp\left(-\frac{3n\nu(H_k)}{104}\right) + L^2 \sum_{k=1}^K \nu(H_k) \exp(-n\nu(H_k)), \quad (27)$$

$$R_n(m_n) \leq 8R(m_n) + 10 \sum_{k=1}^K [\text{osc}_k^{(m)}]^2 \nu(H_k) + \frac{4\sigma^2}{n} \sum_{k=1}^K \exp\left(-\frac{3n\nu(H_k)}{8}\right). \quad (28)$$

Note that the switching-estimates from $R(m_n)$ to $R_n(m_n)$ and from $R_n(m_n)$ to $R(m_n)$ are not symmetric: this reflects that $\nu_n(H_k) > 0$ implies $\nu(H_k) > 0$ but the converse is false. This will become evident in the proof. Putting (27) together with (24) gives a bound (Corollary 4.2) that improves known estimates (like [20, Theorem 11.3] for instance). The improvement comes from the statistical error which is now essentially $K \frac{\sigma^2}{n}$ as soon as the mean number $n\nu(H_k)$ of simulations in each H_k is large enough:

$$L^2 \nu(H_k) \exp(-n\nu(H_k)) \leq L^2 \frac{\sup_{u \geq 0} u e^{-u/2}}{n} \exp(-n\nu(H_k)/2) \ll \frac{\sigma^2}{n}.$$

Corollary 4.2. Assume that m is bounded by $L > 0$, that $\sigma^2 := \sup_{x \in \mathbb{R}^d} \text{Var}(Y \mid X = x) < +\infty$ and that (H_1, \dots, H_K) are disjoint subsets of \mathbb{R}^d , with $\bigcup_k H_k =: D$. For a universal constant $C_{(29)} > 0$, we have

$$R(m_n) \leq C_{(29)} \left[\sum_{k=1}^K [\text{osc}_k^{(m)}]^2 \nu(H_k) + K \frac{\sigma^2}{n} + L^2 \sum_{k=1}^K \nu(H_k) \exp(-n\nu(H_k)) + L^2 \nu(D^c) \right]. \quad (29)$$

Proof of Theorem 4.1. For a function ϕ and subset $A \subset \mathbb{R}^d$, define

$$R(\phi, A) := \mathbb{E} \left[\int_A |m(x) - \phi(x)|^2 \nu(dx) \right], \quad R_n(\phi, A) := \mathbb{E} \left[\int_A |m(x) - \phi(x)|^2 \nu_n(dx) \right].$$

Recall that $m_n = 0$ on D^c ; thus both risks admit the decomposition over disjoint sets

$$\begin{aligned} R(m_n) &= \sum_{k=1}^K \mathbb{E} \left[\int_{H_k} |m(x) - m_n(x)|^2 \nu(dx) \right] + \int_{D^c} |m(x)|^2 \nu(dx) \\ &= \sum_{k=1}^K R(m_n, H_k) + R(0, D^c), \\ R_n(m_n) &= \sum_{k=1}^K \mathbb{E} \left[\int_{H_k} |m(x) - m_n(x)|^2 \nu_n(dx) \right] + \mathbb{E} \left[\int_{D^c} |m(x)|^2 \nu_n(dx) \right] \end{aligned} \quad (30)$$

$$= \sum_{k=1}^K R_n(m_n, H_k) + R(0, D^c) \quad (31)$$

where we have used at the last equality that (X_1, \dots, X_n) are i.i.d. with distribution ν . Thus, it is enough to compare $R(m_n, H_k)$ and $R_n(m_n, H_k)$ for any k .

We start by proving (27). Assume first $\nu_n(H_k) > 0$. Using Cauchy's inequality,

$$\int_{H_k} (m(x) - m_n(x))^2 \nu(dx) \leq 2 \int_{H_k} (m(x) - m_n^*(x))^2 \nu(dx) + 2 \int_{H_k} (m_n^*(x) - m_n(x))^2 \nu(dx). \quad (32)$$

In view of (21), for any $x \in H_k$,

$$|m(x) - m_n^*(x)| \leq \frac{1}{\nu_n(H_k)} \int_{H_k} |m(x) - m(z)| \nu_n(dz) \leq \text{osc}_k^{(m)}. \quad (33)$$

Thus, the first integral on the right hand side of (32) is bounded by $[\text{osc}_k^{(m)}]^2 \nu(H_k)$. For the second integral, observe that, for an arbitrary point $x_k \in H_k$,

$$\int_{H_k} (m_n^*(x) - m_n(x))^2 \nu(dx) = (m_n^*(x_k) - m_n(x_k))^2 \nu(H_k).$$

Combining the above results, it follows that

$$\begin{aligned} \int_{H_k} (m(x) - m_n(x))^2 \nu(dx) &\leq 2[\text{osc}_k^{(m)}]^2 \nu(H_k) + 2(m_n^*(x_k) - m_n(x_k))^2 \nu(H_k) \\ &\leq 2[\text{osc}_k^{(m)}]^2 \nu(H_k) + 4(m_n^*(x_k) - m_n(x_k))^2 \nu_n(H_k) \\ &\quad + 2(m_n^*(x_k) - m_n(x_k))^2 (\nu(H_k) - 2\nu_n(H_k))_+. \end{aligned} \quad (34)$$

For the second term on the right hand side above, we again make use of (33):

$$\begin{aligned} (m_n^*(x_k) - m_n(x_k))^2 \nu_n(H_k) &= \int_{H_k} (m_n^*(x) - m_n(x) \pm m(x))^2 \nu_n(dx) \\ &\leq 2 \int_{H_k} (m_n^*(x) - m(x))^2 \nu_n(dx) + 2 \int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \\ &\leq 2[\text{osc}_k^{(m)}]^2 \nu_n(H_k) + 2 \int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx). \end{aligned}$$

Collecting the different inequalities and taking the expectation in (34) gives (using also $\mathbb{E}[\nu_n(H_k)] = \nu(H_k)$) that

$$\begin{aligned} &\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] \\ &\leq 10[\text{osc}_k^{(m)}]^2 \nu(H_k) + 8\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] \end{aligned}$$

$$+ 2\mathbb{E} [(m_n^*(x_k) - m_n(x_k))^2 (\nu(H_k) - 2\nu_n(H_k))_+ \mathbf{1}_{\nu_n(H_k) > 0}]. \quad (35)$$

It remains to estimate the last expectation term on the right hand side of (35). Write

$$\tilde{m}_n(x) := \arg \min_{\varphi \in \mathcal{K}} \int_{\mathbb{R} \times \mathbb{R}^d} |y - \varphi(x)|^2 \nu_n(dy, dx), \quad (36)$$

so that $m_n(x) = \mathcal{T}_L[\tilde{m}_n(x)]$. Clearly, from (21), m_n^* is bounded by L , therefore $|m_n^*(x) - m_n(x)| \leq |m_n^*(x) - \tilde{m}_n(x)|$. Additionally, on $\{\nu_n(H_k) > 0\}$, from definitions (21) and (36) we have $\tilde{m}_n|_{H_k} = \frac{1}{n\nu_n(H_k)} \sum_{i=1}^n \mathbf{1}_{X_i \in H_k} Y_i$ and $m_n^*|_{H_k} = \frac{1}{n\nu_n(H_k)} \sum_{i=1}^n \mathbf{1}_{X_i \in H_k} m(X_i)$. Denoting by $\mathbb{E}^{(n)}[\cdot] := \mathbb{E}[\cdot | X_1, \dots, X_n]$, we obtain

$$\begin{aligned} \mathbb{E}^{(n)}[(m_n^*(x_k) - m_n(x_k))^2 \mathbf{1}_{\nu_n(H_k) > 0}] &\leq \mathbb{E}^{(n)}[(m_n^*(x_k) - \tilde{m}_n(x_k))^2 \mathbf{1}_{\nu_n(H_k) > 0}] \\ &= \frac{\sum_{i=1}^n \mathbf{1}_{X_i \in H_k} \mathbb{E}^{(m)}[(Y_i - m(X_i))^2]}{n^2 \nu_n^2(H_k)} \mathbf{1}_{\nu_n(H_k) > 0} \\ &\leq \frac{\sigma^2}{n\nu_n(H_k)} \mathbf{1}_{\nu_n(H_k) > 0} \end{aligned} \quad (37)$$

using that sample is independent and that $m(X_i) = \mathbb{E}[Y_i | X_i]$ for all i . Therefore, conditioning inside the expectation, we are left with

$$\begin{aligned} \mathbb{E}[(m_n^*(x_k) - m_n(x_k))^2 (\nu(H_k) - 2\nu_n(H_k))_+ \mathbf{1}_{\nu_n(H_k) > 0}] &\leq \frac{\sigma^2}{n} \mathbb{E} \left[\left(\frac{\nu(H_k)}{\nu_n(H_k)} - 2 \right)_+ \mathbf{1}_{\nu_n(H_k) > 0} \right] \\ &\leq \frac{4\sigma^2}{n} \exp \left(-\frac{3n\nu(H_k)}{104} \right) \end{aligned}$$

where the last inequality follows from Lemma B.1 in Appendix. To summarize, from the above and (35) we have proved

$$\begin{aligned} \mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] &\leq 10[\text{osc}_k^{(m)}]^2 \nu(H_k) \\ &\quad + 8\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] \\ &\quad + \frac{8\sigma^2}{n} \exp \left(-\frac{3n\nu(H_k)}{104} \right). \end{aligned} \quad (38)$$

On the other hand, on $\{\nu_n(H_k) = 0\}$, $m_n|_{H_k} = 0$ and we simply have

$$\begin{aligned} \mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu(dx) \mathbf{1}_{\nu_n(H_k) = 0} \right] &= \int_{H_k} m(x)^2 \nu(dx) (1 - \nu(H_k))^n \\ &\leq L^2 \nu(H_k) \exp(-n\nu(H_k)). \end{aligned} \quad (39)$$

Moreover, $\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \mathbf{1}_{\nu_n(H_k) = 0} \right] = 0$. By summing up (38)-(39) and combining them with (30)-(31), we obtain the announced inequality (27).

We now establish (28). We invert the roles of ν and ν_n in the computations and proceed with the same arguments as before. The inequality (35) becomes

$$\begin{aligned} & \mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] \\ & \leq 10[\text{osc}_k^{(m)}]^2 \nu(H_k) + 8\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu(dx) \mathbf{1}_{\nu_n(H_k) > 0} \right] \\ & \quad + 2\mathbb{E} \left[(m_n^*(x_k) - m_n(x_k))^2 (\nu_n(H_k) - 2\nu(H_k))_+ \mathbf{1}_{\nu_n(H_k) > 0} \right]. \end{aligned}$$

On the other hand, $\mathbb{E} \left[\int_{H_k} (m(x) - m_n(x))^2 \nu_n(dx) \mathbf{1}_{\nu_n(H_k) = 0} \right] = 0$; plugging this and (37) in the above yields

$$R_n(m_n, H_k) \leq 10[\text{osc}_k^{(m)}]^2 \nu(H_k) + 8R(m_n, H_k) + 2\mathbb{E} \left[\frac{\sigma^2}{n\nu_n(H_k)} (\nu_n(H_k) - 2\nu(H_k))_+ \mathbf{1}_{\nu_n(H_k) > 0} \right].$$

Observe here the difference with switching from $R(m_n, H_k)$ to $R_n(m_n, H_k)$ for which we needed to handle an additional term associated to the event $\{\nu_n(H_k) = 0\}$, see (39). To complete the proof, we apply Lemma B.1 and plug this into (31). \square

5 Proof of Theorem 2.6

5.1 Proof of (12)

Define the function $\hat{\psi}_{Y,i}^{(M)}(\cdot)$ (constant on each set $H_k^{(Y,i)}$, $k \in \{1, \dots, K^{(Y,i)}\}$) by

$$\hat{\psi}_{Y,i}^{(M)}(\cdot)|_{H_k^{(Y,i)}} := \frac{1}{\#(A_k^{(Y,i)})} \sum_{m \in A_k^{(Y,i)}} \hat{S}_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \Delta \mathbf{W}^{(i,m)}), \quad (40)$$

if $\#(A_k^{(Y,i)}) > 0$ and 1 otherwise, and $\hat{\psi}_{Y,i}^{(M)}(\cdot)|_{(D^{(Y,i)})^c} = 1$ on the complement of $D^{(Y,i)}$. First, using the 1-Lipschitz property of $\mathcal{T}_{C_y}(\cdot)$ and the triangle inequality, observe that

$$\mathbb{E} \left[\left\| y_i(\cdot) - y_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq \mathbb{E} \left[\left\| y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}_{Y,i}^{(M)}(\cdot)) \right\|_\rho^2 \right]^{1/2} + \mathbb{E} \left[\left\| \hat{\psi}_{Y,i}^{(M)}(\cdot) - \psi_{Y,i}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2}. \quad (41)$$

In what follows, we first estimate the first term of the r.h.s. above using the results from Section 4, then the second term by direct computations.

Let $\mathcal{F}_i^{(M)}$ be the σ -algebra generated by the simulation clouds $\{\mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1}\}$ together with $\{R^{(i,m)} : m = 1, \dots, M_i\}$, and let $\mathbb{E}_i^{(M)}[\cdot]$ be the associated conditional expectation

$\mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{F}_i^{(M)}]$. Then, proceeding as for the proof of Lemma 2.5, we obtain for any $m \in \{1, \dots, M_i\}$ that

$$y_i(R^{(i,m)}) = \mathbb{E}_i^{(M)} \left[\hat{S}_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \Delta \mathbf{W}^{(i,m)}) \right].$$

Thus, $y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}_{Y,i}^{(M)}(\cdot))$ is the difference between the regression function and its empirical approximation, equivalent to the functions m and m_n in Section 4. Therefore, from Corollary 4.2 (working under the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_i^{(M)}]$ in the place of the expectation $\mathbb{E}[\cdot]$, and applying the tower law), we obtain the estimate

$$\mathbb{E} \left[\left\| y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}_{Y,i}^{(M)}(\cdot)) \right\|_{\rho}^2 \right] \leq C_{(29)} \mathcal{E}^{(Y,i)}. \quad (42)$$

We now treat the second term $\mathbb{E} \left[\left\| \hat{\psi}_{Y,i}^{(M)}(\cdot) - \psi_{Y,i}^{(M)}(\cdot) \right\|_{\rho}^2 \right]^{1/2}$ on the r.h.s. of (41). For this we make use of the definitions (11)-(40) and recall that $R^{(i)}$ has the density ρ :

$$\mathbb{E} \left[\left\| \hat{\psi}_{Y,i}^{(M)}(\cdot) - \psi_{Y,i}^{(M)}(\cdot) \right\|_{\rho}^2 \right] = \sum_{k=1}^{K^{(Y,i)}} \mathcal{A}_{k,i} p_k^{(Y,i)} \quad (43)$$

where

$$\begin{aligned} \mathcal{A}_{k,i} &:= \mathbb{E} \left[\left[\#(A_k^{(Y,i)}) \right]^{-2} \left(\sum_{m \in A_k^{(Y,i)}} \mathcal{D}_m \right)^2 \mathbf{1}_{\#(A_k^{(Y,i)}) > 0} \right], \\ \mathcal{D}_m &:= \hat{S}_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \Delta \mathbf{W}^{(m,i)}) - S_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \Delta \mathbf{W}^{(m,i)}). \end{aligned}$$

Now, using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \mathcal{A}_{k,i} &\leq \mathbb{E} \left[\frac{\sum_{m=1}^{M_i} \mathbb{E}_i^{(M)} [\mathcal{D}_m^2] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,m)})}{\#(A_k^{(Y,i)})} \mathbf{1}_{\#(A_k^{(Y,i)}) > 0} \right] \\ &= M_i \mathbb{E} \left[\frac{\mathbb{E}_i^{(M)} [\mathcal{D}_1^2] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)})}{\#(A_k^{(Y,i)})} \mathbf{1}_{\#(A_k^{(Y,i)}) > 0} \right] \\ &= M_i \mathbb{E} \left[\mathbb{E} [\mathcal{D}_1^2 | R^{(i,1)}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1}] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)}) \right] \mathbb{E} \left[\frac{1}{1 + \sum_{m=2}^{M_i} \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,m)})} \right] \\ &\leq \frac{\mathbb{E} [\mathcal{D}_1^2 \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)})]}{p_k^{(Y,i)}} \end{aligned}$$

where the equalities follow because the simulations are i.i.d. and the final inequality follows from direct computation using the binomial distribution (see [20, Lemma 4.1]):

$$\mathbb{E} \left[\frac{1}{1 + \text{Bin}(n, p)} \right] \leq \frac{1}{(n+1)p}.$$

Substituting this back into (43), we obtain

$$\mathbb{E} \left[\left\| \hat{\psi}_{Y,i}^{(M)}(\cdot) - \psi_{Y,i}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq \mathbb{E} [\mathcal{D}_1^2]^{1/2}.$$

Applying the triangle inequality to the \mathbf{L}_2 -norm and making use of the Lipschitz continuity of f , we obtain an estimate on $\mathbb{E} [\mathcal{D}_1^2]^{1/2}$ as follows:

$$\begin{aligned} \mathbb{E} [\mathcal{D}_1^2]^{1/2} &= \mathbb{E} \left[\left| \hat{S}_{Y,i}^{(M)}(\mathbf{X}^{(i)}(h^{(M)}), \Delta \mathbf{W}^{(i)}) - S_{Y,i}^{(M)}(\mathbf{X}^{(i)}(h^{(M)}), \Delta \mathbf{W}^{(i)}) \right|^2 \right]^{1/2} \\ &\leq \sum_{j=i}^{N-1} L_f \mathbb{E} \left[\left| L_{i+1,j+1}^{h^{(M)}}(\mathbf{X}^{(i)}(h^{(M)}), \Delta \mathbf{W}^{(i)}) |y_{j+1}(X_{j+1}^{(i)}(h^{(M)})) - y_{j+1}^{(M)}(X_{j+1}^{(i)}(h^{(M)})) \right|^2 \right]^{1/2} \Delta_j \\ &\leq L_f e^{|h^{(M)}(\cdot)|_\infty^2 T/2} \sum_{j=i}^{N-1} \mathbb{E} \left[|y_{j+1}(X_{j+1}^{(i)}(-h^{(M)})) - y_{j+1}^{(M)}(X_{j+1}^{(i)}(-h^{(M)}))|^2 \right]^{1/2} \Delta_j \end{aligned}$$

where the last equality follows from applying the reverse change of probability measure. The USES property from Proposition 3.1 then yields

$$\mathbb{E} \left[\left\| \hat{\psi}_{Y,i}^{(M)}(\cdot) - \psi_{Y,i}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq \bar{c}_{(16)} L_f e^{C_z^2 T/2} \sum_{j=i}^{N-1} \mathbb{E} \left[\left\| y_{j+1}(\cdot) - y_{j+1}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \Delta_j, \quad (44)$$

where we have used that $|h^{(M)}(\cdot)| \leq C_z$. Substituting (42) and (44) into (41), we conclude that

$$\mathbb{E} \left[\left\| y_i(\cdot) - y_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \leq (C_{(29)} \mathcal{E}^{(Y,i)})^{1/2} + \bar{c}_{(16)} L_f e^{C_z^2 T/2} \sum_{j=i}^{N-1} \mathbb{E} \left[\left\| y_{j+1}(\cdot) - y_{j+1}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \Delta_j.$$

The proof of (12) is now completed by an application of Gronwall's inequality ([18, Proposition 2.4] with $\alpha = \beta = 1/2$). \square

5.2 Proof of (13)

For the computations on Z , we could proceed analogously to Section 5.1 in order to obtain estimates in term of conditional variances. On the other hand, since no specific variance reduction is made, there is no interest for such sophistication. Instead, we follow the error

analysis of [18] for the z -component. It suffices to first use Proposition 3.9 from this reference to estimate the error between the exact \mathbf{L}_2 -norm $\mathbb{E} \left[\left\| z_i(\cdot) - z_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2}$ and the empirical one – like in (26) above – and then to apply the Steps 2 and 3 of the proof of Theorem 3.10 with the conditional variance bounds of Lemma 3.7. This writes

$$\begin{aligned} \mathbb{E} \left[\left\| z_i(\cdot) - z_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} &\leq C_z \sqrt{\frac{2028(K^{(Z,i)} + 1)q \log(3M_i)}{M_i}} \\ &\quad + \sqrt{2} \left(\inf_{\varphi \in \mathcal{K}_{Z,i}} \mathbb{E} \left[|\varphi(R^{(i)}) - z_i(R^{(i)})|^2 \right]^{1/2} + \bar{C}_{Z,i} \sqrt{\frac{K^{(Z,i)}}{M_i}} \right. \\ &\quad \left. + C_M L_f \sum_{k=i+1}^{N-1} \mathbb{E} \left[|y_{k+1}(X_{k+1}^{(i)}(0)) - y_{k+1}^{(M)}(X_{k+1}^{(i)}(0))|^2 \right]^{1/2} \frac{\Delta_k}{\sqrt{t_k - t_i}} \right), \end{aligned}$$

with $\bar{C}_{Z,i} = \frac{C_{(45)}}{\sqrt{T - t_i}}$ (45)

where the constant $C_{(45)}$ depends only on C_g, C_f, L_f, C_M, T, q (see [18, Lemma 3.7] for the explicit expression). Observe that, for all k ,

$$\mathbb{E} \left[|y_{k+1}(X_{k+1}^{(i)}(0)) - y_{k+1}^{(M)}(X_{k+1}^{(i)}(0))|^2 \right]^{1/2} \leq \bar{c}_{(16)} \mathbb{E} \left[\left\| y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2}$$

using the USES property (Proposition 3.1). We have proved the existence of a universal constant $C_{(46)}$ such that

$$\begin{aligned} \mathbb{E} \left[\left\| z_i(\cdot) - z_i^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} &\leq C_{(46)} (\mathcal{E}^{(Z,i)})^{1/2} \\ &\quad + \bar{c}_{(16)} C_M L_f \sum_{k=i+1}^{N-1} \mathbb{E} \left[\left\| y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2} \frac{\Delta_k}{\sqrt{t_k - t_i}}. \end{aligned} \tag{46}$$

By plugging into (46) the estimates (12) on $\mathbb{E} \left[\left\| y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot) \right\|_\rho^2 \right]^{1/2}$, we obtain the announced inequality (13) using [18, Lemma 2.3] with $\alpha = \beta = \gamma = 1/2$. \square

6 Numerical experiments

We consider the Brownian motion model $X = W$ ($d = q$). Under the optimal change of probability measure, this process becomes a Brownian motion with drift:

$$d\tilde{X}_t = \frac{z(t, \tilde{X}_t)^\top}{y(t, \tilde{X}_t)} dt + dW_t; \tag{47}$$

we shall approximate this process using an Euler scheme and approximate the drift as in Algorithm 2.1.

For USES, we simulate using the random variable R with Laplace distribution, whose density is given in (18) with μ to be specified.

In this section, we trivially extend (\mathbf{H}_C) in this section to $Y \geq \gamma > 0$ for general γ rather than $\gamma = 1$.

For the least-squares Monte Carlo, we use two different sets of basis functions: for the approximation of $y_i(\cdot)$, a uniform hypercubes basis defined on the domain $D^{(Y,i)} := [-L_y/\mu, L_y/\mu]^q$ for $L_y > 0$ to be specified; for the approximation of $z_i(\cdot)$, we use polynomials of degree less than or equal to $\kappa \geq 1$ on the domain $D^{(Z,i)} := [-L_z/\mu, L_z/\mu]^q$ for $L_z > 0$ to be specified, and zero on $(D^{(Z,i)})^c$, i.e. the basis functions take the form

$$p_{l_1, \dots, l_q}(x) = \mathbf{1}_{D^{(Z,i)}}(x) \prod_{j=1}^q x_j^{l_j}, \quad x = (x_1, \dots, x_q), \quad \text{on each coordinate of } z_i,$$

where (l_1, \dots, l_q) satisfy $l_j \in \mathbb{N}_+ \cup \{0\}, \forall j$ and $\sum_{j=1}^q l_j \leq \kappa$.

The number of basis functions K and the number of simulations M are equal on every time point and parameterized according to the number of time-steps N : To assess the performance of the algorithm, we compute the average mean squared error (MSE) over 10^3 independent runs of the algorithm for three error statistics:

$$\begin{aligned} MSE_{Y, \max} &:= 10^{-3} \max_{0 \leq i \leq N-1} \sum_{m=1}^{10^3} |y_i(R_m) - y_i^{(M)}(R_m)|^2, \\ MSE_{Y, \text{av}} &:= 10^{-3} N^{-1} \sum_{m=1}^{10^3} \sum_{i=1}^{N-1} |y_i(R_m) - y_i^{(M)}(R_m)|^2, \\ MSE_{Z, \text{av}} &:= 10^{-3} N^{-1} \sum_{m=1}^{10^3} \sum_{i=1}^{N-1} |z_i(R_m) - z_i^{(M)}(R_m)|^2, \end{aligned} \tag{48}$$

where the simulations $\{R_m; m = 1, \dots, 10^3\}$ are i.i.d. and independently drawn from those simulations used for the LSMC scheme.

In addition, we run experiments with an idealized version of Algorithm 2.1, which we term the Semi-Perfect ISMWDP (SPISMWDP). This algorithm assumes that, in the computations for the Y component, one does not need to approximate the functions $z_i(\cdot)$; as such, it is not fully implementable. It serves to illustrate the maximal scope of variance reduction that the ISMWDP algorithm can potentially deliver. In comparison with Algorithm 2.1, SPISMWDP is defined using z_i in place of $z_i^{(M)}$, everything else being the same.

The algorithm is implemented in C++. It is run on a 64bit Linux server, running Ubuntu 12.04.3. It has 12 cores (although no parallel computation is made), each being 2.9GHz. The compilation makefile is built through CMake, and compiled with gcc version

4.8 and optimized using the `-Os` flag. BLAS and LAPACK operations are provided by the native system libraries, and we can expect vectorization also through compiler optimization.

6.1 Example 1: approximating an oscillating solution

Let $\mathbf{1}_q$ be the q -dimensional vector with components all 1. For $\gamma > 0$ and $\lambda > 0$, let $g(x) := 1 + \gamma + \sin(\lambda \mathbf{1}_q^\top x)$ and

$$f(t, x, y) := 1 \wedge \left(y - \gamma - 1 - \frac{\sin(\lambda \mathbf{1}_q^\top x)}{\exp(\lambda^2 q(T-t)/2)} \right)^2.$$

The solution to the continuous time BSDE is given by

$$y(t, x) = 1 + \gamma + \frac{\sin(\lambda \mathbf{1}_q^\top x)}{\exp(\lambda^2 q(T-t)/2)}, \quad z_k(t, x) = \frac{\lambda \cos(\lambda \mathbf{1}_q^\top x)}{\exp(\lambda^2 q(T-t)/2)}.$$

In what follows, we take $T = 1$ and $\gamma = 0.6$ (the lower bound in (\mathbf{H}_C) is replaced by $\gamma > 0$, the algorithm and the convergence analysis can be easily adjusted to this). In order to maximize $z_i(x)/y_i(x)$ at $x = 0$, we choose $\lambda = 1/\sqrt{q}$. This is to make the drift impact large at the starting point of each path simulation, where the highest probability occupation region is around zero. In this way, we hope to ascertain significantly different results between ISMWDP and MWDP.

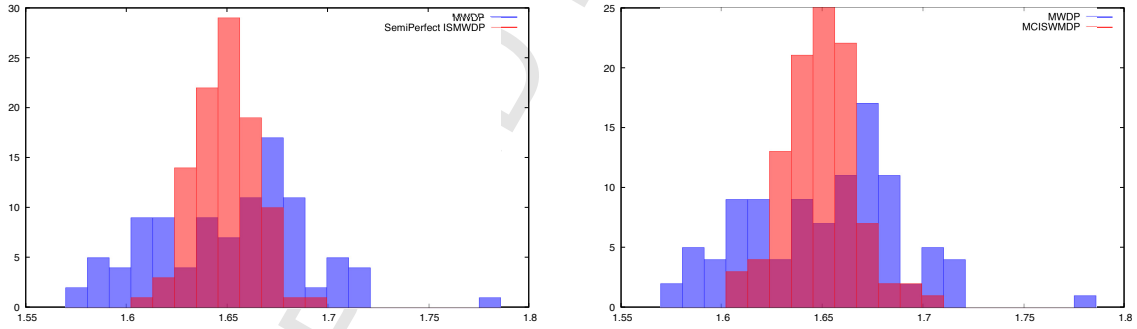


Figure 1: Two dimensional example: left is SPISMWDP and right is ISMWDP, in both cases compared against MWDP. $N = 10$, $M = 6000$. The reference value to compare against is $y(0, 0) = 1.6$.

We set $\mu = \lambda$ for the USES random variable. For the uniform hypercubes basis, we set the domain boundary using $L_y = 5$ and the number of uniform hypercubes used for each point of the N -point time-grid to be $K_N = (11 + 3\sqrt{N})^q$. For the polynomial basis, we set the domain boundary using $L_z = 4$, and the maximal polynomial degree $\kappa = 6$ to approximate $z_i(\cdot)$. This means that in dimension $q = 2$ there are 28 basis functions approximating each coordinate

of $z_i(\cdot)$. The number of simulations used for each point of the N -point time-grid is given by $M_N = 4.3 \times K_N \times \sqrt{N}$.

In Figure 1 and Tables 1-2-3, we report experiments in dimension $q = 2$. The histograms of Figure 1 for the solution y at point $(t, x) = (0, 0)$ illustrate the variance reduction effect very clearly: the distribution of the solution $y_0(0)$ is more concentrated around its mean using ISMWDP than MWDP. Because the reference value is 1.6, it also shows that the approximation bias (due to approximation on basis functions) is a significant component of the statistical error. Tables 1-2-3 report the global errors measured as in (48). One can observe improvements in the error terms Av Y for ISMWDP against MWDP, but the improvement is modest due to the relatively significant bias. This error improvement is increasing with the number of time steps. The SPISMWDP scheme shows more substantial improvements, which we would expect because we use the exact value of z_i .

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	2778	0.07303	0.0639026	0.086021	0.117
10	5996	0.055779	0.0446373	0.0616438	0.8395
15	8809	0.0504425	0.0379929	0.0512298	2.7745
20	12018	0.0454005	0.0324522	0.0450157	6.9215

Table 1: **ISMWDP**, $q = 2$

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	2778	0.0677722	0.0559159	0.0880097	0.0695
10	5996	0.0509774	0.0388176	0.0605647	0.3375
15	8809	0.0461908	0.0322247	0.0515244	0.9735
20	12018	0.0434087	0.0289556	0.0437723	2.163

Table 2: **SPISMWDP**, $q = 2$

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	2778	0.076381	0.0649131	0.0898902	0.046
10	5996	0.0583666	0.0460254	0.0608955	0.2475
15	8809	0.0494551	0.0381727	0.0510557	0.654
20	12018	0.0476445	0.0348098	0.045483	1.4205

Table 3: **MWDP**, $q = 2$

6.2 Example 2: diminishing Z example

Define the function $\omega(t, x) = \exp(t + \mathbf{1}_q^\top x)$. We perform numerical experiments on the BSDE with data $g(x) = \gamma + \omega(T, \lambda x)(1 + \omega(T, \lambda x))^{-1}$ and

$$\frac{f(t, x, y) = \omega(t, \lambda x)}{(1 + \omega(t, \lambda x))^2 (q\lambda^2(y - \gamma) - 1 - \frac{q}{2}\lambda^2)}.$$

We shall work with $T = 1$, $\gamma = 0.6$ and $q = 2$. The continuous-time BSDE has explicit solutions in this framework, given by

$$y(t, x) = \gamma + \omega(t, \lambda x)(1 + \omega(t, \lambda x))^{-1}, \quad z_k(t, x) = \lambda \omega(t, \lambda x)(1 + \omega(t, \lambda x))^{-2},$$

where $z_k(t, x)$ is the k -th component of the q -dimensional cylindrical function $z(t, x) \in (\mathbb{R}^q)^\top$. It is clear that $z(t, x) \rightarrow 0$ exponentially fast as $|x| \rightarrow \infty$.

A good choice for λ to demonstrate the difference between ISMWDP and MWDP is not so clear as in the previous subsection. Since $z(t, x)/y(t, x) = \lambda \mathbf{1}_q \frac{\omega}{(1+\omega)(\gamma(1+\omega)+\omega)}(t, \lambda x) > 0$ is the optimal drift, choosing a large λ would increase the drift around $x = 0$, but quickly push the process \tilde{X} of (47) into a region where z_i and y_i are not well approximated, leading to large error due to drift mis-specification. On the other hand, small λ implies that the drift is very small, so one does not anticipate significant improvements from ISMWDP. We use $\lambda = 2$.

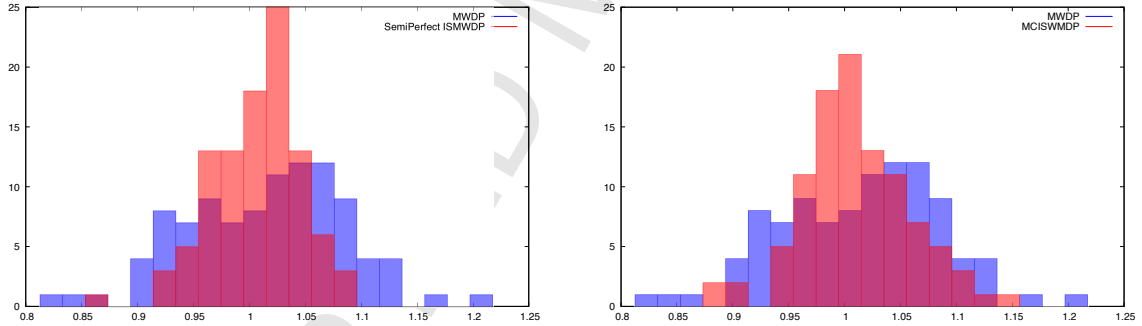


Figure 2: Two dimensional example: left is SPISMWDP and right is ISMWDP, in both cases compared against MWDP. $N = 10$, $M = 3000$. The reference value to compare against is $y(0, 0) = 1.1$.

We set $\mu = \lambda$ for the USES random variable. For the hypercubes basis, we set the domain boundary to $L_y = 4.5$ and the number of uniform cubes used at each point of the N -point time-grid to be $K_N = (11 + 3\sqrt{N})^q$. For the polynomials basis, we set the outer boundary to be $L_z = 3$ and the maximal polynomial degree to be $\kappa = 4$; this means that in dimension $q = 2$ there are 15 basis functions for the approximation of each coordinate of $z_i(x)$. The number of simulations used for each point of the N -point time-grid is given by $M_N = 2.2 \times K_N \times \sqrt{N}$.

The results on the error terms (48) from the experiments are illustrated in Tables 4-5-6. In particular, one can observe improvements in the error terms Av Y for ISMWDP against MWDP (as before, more substantial for SPISMWDP). The improvement increases with the number of time steps. The histograms of Figure 2 illustrate the variance reduction effect in the computation of $y_0(0)$, showing also that the bias of the approximation has a role to play in the error evaluation. Observe that the impact of variance reduction for this example is somewhat weaker than in Section 6.1.

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	1421	0.0485281	0.0379537	0.0635861	0.0515
10	3068	0.039387	0.027375	0.0426894	0.248
15	4507	0.0359338	0.0242747	0.039852	0.826
20	6149	0.0343716	0.022069	0.0373225	1.9995

Table 4: **ISMWDP**, $q = 2$

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	1421	0.042808	0.0346473	0.0625695	0.037
10	3068	0.0371875	0.0258433	0.0440036	0.136
15	4507	0.0347329	0.0226123	0.0406991	0.34
20	6149	0.0338109	0.0210026	0.0380248	0.806

Table 5: **SPISMWDP**, $q = 2$

Time points (N)	Simulations (M_N)	Max Y	Av Y	Av Z	Comp. time (s)
5	1421	0.049516	0.0389116	0.0664352	0.031
10	3068	0.0420609	0.0290501	0.0448	0.0825
15	4507	0.0396252	0.0258874	0.039865	0.2365
20	6149	0.0365294	0.0232032	0.0370308	0.55

Table 6: **MWDP**, $q = 2$

6.3 Concluding remarks on the numerical experiments

The experiments of the first three subsections have shown that the statistical error is indeed significantly reduced using a ISMWDP scheme, up to the point that the approximation error due to basis functions dominates the global error.

We have not discussed computational time so far. It is clear from all the tables that the computational time for MWDP is lower than the other schemes given the same numerical set up, especially against the fully implementable importance sampling scheme. The reason

for this is that the MWDP scheme does not make any computations on the function z_i . This is a considerable reduction of computational time, especially as the dimension q grows and correspondingly the dimension of the basis for approximating z_i grows. Highlighting the computational time may be not the most important issues when solving BSDEs: in some situations (memory constraint in large dimension [19] or in parallel computing [17], calibration on small data [16]), the number of paths to be used is imposed and the objectives become to obtain the most efficient algorithm with a given Monte Carlo simulation effort. In that case, the use of ISMWDP type scheme for reducing statistical error is certainly advantageous.

The above experiments have shown there is scope for improvement of the ISMWDP scheme. It is clear that improving the estimation of the function z_i leads to significantly better performance of the variance reduction scheme, as illustrated by the results for the SPISMWDP scheme. We shall consider these improvements in future works.

A Distributions satisfying (H_ρ) : proof of Proposition 3.3

Observe that thanks to the product form of the densities (18)-(19)-(20) (due to the independence of the coordinates), the d -dimensional result follows from the case $d = 1$. Thus we shall only prove the one-dimensional result.

Case (a). For $r \in \mathbb{R}$, set $\mathcal{I}(r) := \int_{\mathbb{R}} e^{r|z|} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz$. Then a direct triangle inequality gives

$$\rho(y)\mathcal{I}(-\mu\sqrt{\lambda}) \leq \int_{\mathbb{R}} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz \leq \rho(y)\mathcal{I}(\mu\sqrt{\lambda}).$$

Clearly, $\mathcal{I}(\cdot)$ is a positive and continuous function, thus bounded from below and from above on the compact $[-\mu\sqrt{\lambda}, \mu\sqrt{\lambda}]$. \square

Case (b). We prove only the case $\mu = 1$; the general case $\mu > 0$ is similar and is left to the reader. Set

$$\mathcal{J}(y, \lambda) := \int_{\mathbb{R}} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz. \quad (49)$$

First, observe that $\mathcal{J}(\cdot)$ and $\rho(\cdot)$ are both positive and continuous: hence, for any given $y_0 > 0$, $\mathcal{J}(y, \lambda)/\rho(y)$ is bounded from above and from below uniformly on $[-y_0, y_0] \times [0, \Lambda]$. Fixing $y_0 = 1$, it remains to check (14) only for $(y, \lambda) \in [-y_0, y_0]^c \times [0, \Lambda]$.

Upper bound. Write $\mathcal{J} := \mathcal{J}_1 + \mathcal{J}_2$ where \mathcal{J}_1 and \mathcal{J}_2 correspond respectively to the integral on $A_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|/2\}$ and on $[A_\lambda]^c$. On the one hand on A_λ , use $(1 + |y + z\sqrt{\lambda}|) \geq (1 + |y|/2) \geq \frac{1}{2}(1 + |y|)$ to get $\mathcal{J}_1(y, \lambda) \leq 2^{k+1}\rho(y)$. On the other hand, obviously we have

$$\mathcal{J}_2(y, \lambda) \leq \frac{k}{2} \int_{[A_\lambda]^c} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz \leq k\mathcal{N}\left(-\frac{|y|}{2\sqrt{\lambda}}\right) \leq c(k, \Lambda)\rho(y)$$

where $\mathcal{N}(\cdot)$ is the cumulative probability function of the standard normal distribution. $c(k, \Lambda) > 0$ depends only on k and Λ and ensures the last inequality is valid for any $|y| > y_0 := 1$, because the density ρ decreases to 0 much slower than the Gaussian density.

Lower bound. By integrating only on $B_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|\}$ and using $(1 + |y + z\sqrt{\lambda}|) \leq (1 + 2|y|) \leq 2(1 + |y|)$ on that set, we obtain

$$\begin{aligned} \mathcal{J}(y, \lambda) &\geq \left(\frac{k}{2}\right)(1 + |y|)^{-k-1}2^{-k-1} \int_{B_\lambda} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz \geq 2^{-k} \rho(y) \left(\frac{1}{2} - \mathcal{N}\left(-\frac{|y|}{\sqrt{\Lambda}}\right)\right) \\ &\geq 2^{-k} \rho(y) \left(\frac{1}{2} - \mathcal{N}\left(-\frac{1}{\sqrt{\Lambda}}\right)\right) \end{aligned}$$

for any $|y| \geq |y_0| := 1$ and $\lambda \in [0, \Lambda]$. \square

Case (c). As for the case (b), we give the proof only for $\mu = 1$, the general case being analogous. Using the same arguments as before, (14) easily holds for $(y, \lambda) \in [-1, 1] \times [0, \Lambda]$ and it remains to prove it for $|y| > 1$ and $\lambda \in [0, \Lambda]$. We define \mathcal{J} as in (49), but with the density ρ defined in (20).

Upper bound. Write $\mathcal{J} := \mathcal{J}_1 + \mathcal{J}_2$ where \mathcal{J}_1 and \mathcal{J}_2 correspond respectively to the integral on $A_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|/2\}$ and on its complement. On A_λ , use $(1 + |y + z\sqrt{\lambda}|) \geq \frac{1}{2}(1 + |y|)$ and $(1 + |y + z\sqrt{\lambda}|)^{2/\alpha} \geq (1 + |y| - |z|\sqrt{\lambda})^{2/\alpha} \geq (1 + |y|)^{2/\alpha} - (|z|\sqrt{\lambda})^{2/\alpha}$ (owing to $2/\alpha < 1$), to get

$$\begin{aligned} \mathcal{J}_1(y, \lambda) &\leq \int_{A_\lambda} \frac{e}{\alpha} e^{-(1+|y|)^{2/\alpha}} e^{(|z|\sqrt{\lambda})^{2/\alpha}} (1 + |y|)^{2/\alpha-1} 2^{1-2/\alpha} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz \\ &\leq \rho(y) 2^{1-2/\alpha} \int_{\mathbb{R}} e^{(|z|\sqrt{\Lambda})^{2/\alpha}} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz. \end{aligned}$$

Using the same arguments as for the case (b), we show that, for all $|y| \geq |y_0| := 1$, $\mathcal{J}_2(y, \lambda) \leq c(\alpha, \Lambda) \rho(y)$ for some constant $c(\alpha, \Lambda) > 0$ depending only on α and Λ .

Lower bound. It is obtained by integrating only on $B'_\lambda := \{z : |z|\sqrt{\lambda} \leq 1\} \cap B'_\Lambda$. On B'_λ , since $|y| > 1$ we have $|z|\sqrt{\lambda} \leq |y|$, and therefore $(1 + |y + z\sqrt{\lambda}|) \leq 2(1 + |y|)$ and $(1 + |y + z\sqrt{\lambda}|)^{2/\alpha} \leq (1 + |y|)^{2/\alpha} + (|z|\sqrt{\Lambda})^{2/\alpha}$. We then deduce

$$\begin{aligned} \mathcal{J}(y, \lambda) &\geq \int_{B'_\lambda} \frac{e}{\alpha} e^{-(1+|y|)^{2/\alpha}} e^{-(|z|\sqrt{\Lambda})^{2/\alpha}} (1 + |y|)^{2/\alpha-1} 2^{2/\alpha-1} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz \\ &\geq \rho(y) 2^{2/\alpha-1} \int_{B'_\Lambda} e^{-(|z|\sqrt{\Lambda})^{2/\alpha}} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} dz. \end{aligned}$$

The proof is complete. \square

B Large deviation estimates for binomial distribution

Lemma B.1. *Let X be a random variable with distribution $\text{Bin}(n, p)$ with $p \in [0, 1]$ and $n \geq 1$. Then,*

$$\mathbb{E} \left[\left(\frac{np}{X} - 2 \right)_+ \mathbf{1}_{X>0} \right] \leq 4 \exp \left(-\frac{3np}{104} \right), \quad (50)$$

$$\mathbb{E} \left[\left(1 - \frac{2np}{X} \right)_+ \mathbf{1}_{X>0} \right] \leq 2 \exp \left(-\frac{3np}{8} \right). \quad (51)$$

Proof. If $p = 0$ (respectively $p = 1$), then $X = 0$ (resp. $X = n$) a.s. and the above inequalities are obvious. Assume from now on that $p(1-p) > 0$.

We start by proving (50). Firstly, observe that X and $(npX^{-1} - 2)$ are both positive if and only if $0 < X < np/2$. Therefore, denoting by X' the random variable with distribution $\text{Bin}(n+1, p)$, we have

$$\begin{aligned} \mathcal{I} := \mathbb{E} \left[\left(\frac{np}{X} - 2 \right)_+ \mathbf{1}_{X>0} \right] &= \sum_{i=1}^{\lfloor np/2 \rfloor} \left(\frac{np}{i} - 2 \right) \binom{n}{i} p^i (1-p)^{n-i} \\ &\leq \sum_{i=1}^{\lfloor np/2 \rfloor} \frac{n(i+1)}{(n+1)i} \times \frac{(n+1)!}{(i+1)!(n+1-i-1)!} p^{i+1} (1-p)^{(n+1)-(i+1)} \\ &\leq 2\mathbb{P}(2 \leq X' \leq np/2 + 1) \\ &\leq 2\mathbb{P}(X' - (n+1)p \leq np/2 + 1 - (n+1)p) \\ &= 2\mathbb{P}(X' - \mathbb{E}[X'] \leq 1 - p - np/2). \end{aligned}$$

Now, assuming that $1 - p - np/2$ is smaller than $-(n+1)p/4$, i.e. $np/4 \geq 1 - 3p/4$, one can apply Bernstein's inequality [20, Lemma A.2] above to determine

$$\begin{aligned} \mathcal{I} &\leq 2\mathbb{P} \left(\frac{X' - \mathbb{E}[X']}{n+1} \leq -\frac{p}{4} \right) \leq 4 \exp \left(-\frac{(n+1)p^2/16}{2p(1-p) + 2p/12} \right) \\ &\leq 4 \exp \left(-\frac{3np}{8(13-12p)} \right) \leq 4 \exp \left(-\frac{3np}{104} \right). \end{aligned}$$

In particular, the inequality is valid for $n \geq 4/p$. On the other hand, for $n \leq 4/p$, observe that

$$\mathcal{I} \leq \mathbb{E} \left[\left(\frac{4}{X} - 2 \right)_+ \mathbf{1}_{X>0} \right] \leq 2 \leq 2 \exp \left(\frac{3}{26} \right) \exp \left(-\frac{3np}{104} \right) \leq 4 \exp \left(-\frac{3np}{104} \right).$$

Thus, (50) is proved.

Now we justify (51). We simply observe that

$$\mathbb{E} \left[\left(1 - \frac{2np}{X} \right)_+ \mathbf{1}_{X>0} \right] \leq \mathbb{P} \left(\frac{X - \mathbb{E}[X]}{n} > p \right) \leq 2 \exp \left(-\frac{np^2}{2p(1-p) + 2p/3} \right)$$

using the Bernstein inequality, which gives (51) after simplification. \square

References

- [1] S. Alanko and M. Avellaneda. Reducing variance in the numerical solution of BSDEs. *Comptes Rendus Mathématique*, 351(3):135–138, 2013.
- [2] V. Bally and A. Matoussi. Weak solutions for SPDEs and Backward Doubly Stochastic Differential Equations. *Journal of Theoretical Probability*, 14:125–164, 2001.
- [3] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.*, 41(3B):1831–1863, 2013.
- [4] T. Ben Zineb and E. Gobet. Preliminary control variates to improve empirical regression methods. *Monte-Carlo methods and Applications*, 19(4):331–354, 2013.
- [5] C. Bender and T. Moseler. Importance sampling for backward SDEs. *Stochastic Analysis and Applications*, 28:226–253, 2010.
- [6] C. Bender and J. Steiner. Least-squares Monte Carlo for BSDEs. In R. Carmona, P. Del Moral, P. Hu, and N. Oudjane, editors, *Numerical Methods in Finance*, pages 257–289. Series: Springer Proceedings in Mathematics, Vol. 12, 2012.
- [7] J. Blanchet, P. Glynn, and Y. Shi. On Lyapunov inequalities and subsolutions for efficient importance sampling. *ACM TOMACS Transactions in Modeling and Computer Simulation*, 22(3):13, 2012.
- [8] P. Briand and C. Labart. Simulation of BSDEs by Wiener chaos expansion. *Annals of Applied Probability*, 24(4):1120–1171, 2014.
- [9] J.A. Bucklew. *Introduction to Rare Event Simulation*. Springer, 2004.
- [10] J.-F. Chassagneux and D. Crisan. Runge-Kutta schemes for backward stochastic differential equations. *The Annals of Applied Probability*, 24(2):679–720, 2014.
- [11] S. Crepey. *Financial modeling: a Backward Stochastic Differential Equations perspective*. Springer. Springer-Verlag, Berlin, 2013.

- [12] N. El Karoui, S. Hamadène, and A. Matoussi. Backward stochastic differential equations and applications. In R. Carmona, editor, *Indifference pricing: theory and applications*, chapter 8, pages 267–320. Springer-Verlag, 2008.
- [13] A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. *Ann. Appl. Probab.*, 21(4):1322–1364, 2011.
- [14] E. Gobet and C. Labart. Solving BSDE with adaptive control variate. *SIAM Numerical Analysis*, 48(1):257–277, 2010.
- [15] E. Gobet, J-P. Lemor, and X. Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. of Appl. Probab.*, 15(3), 2172–2202, 2005.
- [16] E. Gobet, G. Liu and J. Zubelli. A non-intrusive stratified resampler for regression Monte Carlo: application to solving non-linear equations. *Preprint hal-01291056*, 2016.
- [17] E. Gobet, J. Lopez-Salas, P. Turkedjiev and C. Vasquez. Stratified regression Monte-Carlo scheme for semilinear PDEs and BSDEs with large scale parallelization on GPUs. *In revision for SIAM Journal of Scientific Computing*, preprint hal-01186000, 2015.
- [18] E. Gobet and P. Turkedjiev. Approximation of backward stochastic differential equations using Malliavin weights and least-squares regression. *Bernoulli*, 22(1):530–562, 2016.
- [19] E. Gobet and P. Turkedjiev. Linear regression MDP scheme for discrete backward stochastic differential equations under general conditions. *Math. Comp.*, 85(299):1359–1391, 2016.
- [20] L. Györfi, M. Kohler, A. Krzyżak, and H. Walk. *A distribution-free theory of nonparametric regression*. Springer Series in Statistics, 2002.
- [21] Y. Hu, D. Nualart, and X. Song. Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *Ann. Appl. Probab.*, 21(6):2379–2423, 2011.
- [22] B. Jourdain and J. Lelong. Robust adaptive importance sampling for normal random vectors. *Ann. Appl. Probab.*, 19(5):1687–1718, 2009.
- [23] N. Kazamaki. Continuous exponential martingales and BMO, volume 1579 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1994.
- [24] V Lemaire and S. Menozzi. On some non asymptotic bounds for the euler scheme. *Electron. J. Probab.*, 15:no. 53, 1645–1681, 2010.

- [25] V. Lemaire and G. Pagès. Unconstrained recursive importance sampling. *Ann. Appl. Probab.*, 20(3):1029–1067, 2010.
- [26] J-P. Lemor, E. Gobet, and X. Warin. Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli*, 12(5), 889–916, 2006.
- [27] J. Ma and J. Yong. *Forward-Backward Stochastic Differential Equations*. Lecture Notes in Mathematics, 1702, Springer-Verlag, 1999. A course on stochastic processes.
- [28] J. Ma and J. Zhang. Representation theorems for backward stochastic differential equations. *The Annals of Applied Probability*, 12(4):1390–1418, 2002.
- [29] T. Moseler. A Picard-type iteration for backward stochastic differential equations: convergence and importance sampling. Ph.D. thesis, Fachbereich Mathematik und Statistik der Universität Konstanz, 2010.
- [30] N.J. Newton. Variance reduction for simulated diffusions. *SIAM Journal on Applied Mathematics*, 54(6):1780–1805, 1994.
- [31] G. Rubino and B. Tuffin. *Rare Event Simulation using Monte Carlo Methods*. Wiley, 2009.
- [32] R.Y. Rubinstein and D.P. Kroese. *Simulation and the Monte Carlo method*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2008.
- [33] J. Smoller. *Shock waves and reaction-diffusion equations*, volume 258 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, New York, second edition, 1994.
- [34] P. Turkedjiev. Two algorithms for the discrete time approximation of Markovian backward stochastic differential equations under local conditions. *Electronic Journal of Probability*, 20(50), 2015.